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Computation of Nash equilibria: a gradient-type and relaxation-type method

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Abstract: We present two computational methods for determining the Nash equilibrium points for a particular class of games. The analysis is based on the concepts of descent directions, weak and strong Nash stationary points. An illustrative numerical example is shown at the end of the paper.

A Introduction

Traditional game theory takes as its basic distinction that between cooperative games and noncooperative games. In cooperative games, the players are assumed to be free to communicate in any way they choose before and during the game. More importantly, they are also assumed to be able to bind themselves to any agreements that may be reached during such bull sessions. A noncooperative game ought properly to be defined as a game that is not cooperative. More often, the terminology is used to signify a game in which agreements are never binding on the players.

The notion of Nash equilibrium of an n-person noncooperative game has an important significance in game theory and economic applications. The existence of this point has been extensively studied, beginning with the important existence theorem due to Nash ([16], [17]) and continuing with other several investigations ([2], [19]). From another point of view, practical algorithms for finding Nash equilibrium have been elaborated only recently. These numerical algorithms seem to be important in real-word applications, where the players do not know each other's objective functions and other relevant information ([6]). The players only have their own tentative decisions to communicate to each other during each phase of the computation as seen in [4] and [7]. Among the numerical algorithms formulated so far, we may focus on those presented in [14], [6], [5], [18].

In this paper, we consider a class of games and we investigate two numerical methods for computing the Nash equilibrium. The numerical methods presented here start from the ideas of Nash stationary points ([13]) and computation of Nash equilibria via parallel gradient descent ([6]).

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B The abstract framework

The purpose of this section is to recall some basic facts concerning our topic.

Let us consider X, Y two Hausdorff topological vector spaces, as well as the subsets $K_1 \subseteq X$, $K_2 \subseteq Y$.

Consider a non-cooperative game of two players: the first players task is to choose a strategy x from K_1, and the second players task is to choose a strategy y from K_2.

The traditional model for the game theory classifies the strategies of each player using a loss function (payoff function). Denote by $f : K_1 \times K_2 \to \Re$ the loss function associated with the first player and by $g : K_1 \times K_2 \to \Re$ the loss function associated with the second player. It is obvious that each player makes decisions in order to minimize his loss. For this reason, we are led to the following definition:

Definition B.1 *The point $(a, b) \in K_1 \times K_2$ is called* **a Nash equilibrium** *point for the game if:*

$$\begin{array}{l} f(a,b) \le f(x,b) \ , \forall x \in K_1 \\ g(a,b) \le g(a,y) \ , \forall y \in K_2 \end{array} .$$

These inequalities assert that the strategies (a, b) are optimal responses of each player, if the strategy of the partner is unchanged ([1]).

J. Nash proved the following existence theorem([16], [17], [2]):

Theorem B.2 *Suppose that K_1, K_2 are nonempty, compact and convex. Suppose that the payoffs f, g are continuous functions with*

$$\begin{array}{l} \forall y \in K_2 \text{ fixed, the function } x \in K_1 \longmapsto f(x,y) \text{ is quasiconvex} \\ \forall x \in K_1 \text{ fixed, the function } y \in K_2 \longmapsto g(x,y) \text{ is quasiconvex} \end{array} .$$

Then at least one Nash equilibrium exists.

Remark B.3 *The above result offers a sufficient condition for the existence of an equilibrium point, however equilibrium points may exist even if the conditions of the theorem are not satisfied.*

In [13] there are presented other concepts which are close related to the Nash equilibrium points. Let us consider X, Y two real normed spaces, and K_1, K_2 nonempty compact convex subsets of X and Y, respectively.

We say that the real-valued function **f** belongs to the class $\mathcal{L}_1(K_1 \times K_2)$ if the following three properties hold:

(i) there exists a convex open set D_1 such that $K_1 \subseteq D_1 \subseteq X$ and f is defined on $D_1 \times K_2$;

(ii) for all $(x, y) \in K_1 \times K_2$, there exists $L > 0$ and a neighborhood U of (x, y) such that

$$|f(x', z) - f(x'', z)| \le L \|x' - x''\|, \forall (x', z), (x'', z) \in U \cap (D_1 \times K_2);$$

(iii) f is continuous on $K_1 \times K_2$.

Similarly, **g** $\in \mathcal{L}_2(K_1 \times K_2)$ if:

(i) there exists a convex open set D_2 such that $K_2 \subseteq D_2 \subseteq Y$ and g is defined on $K_1 \times D_2$;

(ii) for all $(x, y) \in K_1 \times K_2$,there exists $L > 0$ and a neighborhood U of (x, y) such that

$$|g(z, y') - g(z, y'')| \le L \|y' - y''\|, \forall (z, y'), (z, y'') \in U \cap (K_1 \times D_2);$$

(iii) g is continuous on $K_1 \times K_2$.
Let us denote the cones

$$\begin{aligned} S_{K_1}(x) &= \{\lambda(x'-x) : x' \in K_1, \lambda > 0\} \\ S_{K_2}(y) &= \{\lambda(y'-y) : y' \in K_2, \lambda > 0\} \end{aligned}$$

as well as the contingent cones

$$\begin{aligned} T_{K_1}(x) &= \{h_1 \in X : \exists \varepsilon_k \to 0^+, \exists h_k \to h_1 \text{ s.t. } x + \varepsilon_k h_k \in K_1, \forall k \in N\} \\ T_{K_2}(y) &= \{h_2 \in Y : \exists \varepsilon_k \to 0^+, \exists h_k \to h_2 \text{ s.t. } y + \varepsilon_k h_k \in K_2, \forall k \in N\}. \end{aligned}$$

If K_1, K_2 are convex sets, then according to [3], we have

$$T_{K_1}(x) = clS_{K_1}(x) \quad , \quad T_{K_2}(y) = clS_{K_2}(y).$$

Then, by [13], the **strong partial directional derivatives** are defined as:

$$\begin{aligned} D_1^s f(a,b)(h_1) &: = \lim \inf_{\substack{\varepsilon \to 0^+ \\ a+\varepsilon h_1 \in K_1}} \frac{f(a+\varepsilon h_1, b) - f(a,b)}{\varepsilon}, \ h_1 \in S_{K_1}(a) \\ D_2^s g(a,b)(h_2) &: = \lim \inf_{\substack{\varepsilon \to 0^+ \\ b+\varepsilon h_2 \in K_2}} \frac{g(a, b+\varepsilon h_2) - g(a,b)}{\varepsilon}, \ h_2 \in S_{K_2}(b) \end{aligned}$$

and **weak partial directional derivatives** are defined as:

$$\begin{aligned} D_1^w f(a,b)(h_1) &: = \lim \sup_{\substack{\varepsilon \to 0^+ \\ x \to a \\ y \to b, y \in K_2}} \frac{f(x+\varepsilon h_1, y) - f(x,y)}{\varepsilon}, \ h_1 \in X \\ D_2^w g(a,b)(h_2) &: = \lim \sup_{\substack{\varepsilon \to 0^+ \\ x \to a, x \in K_1 \\ y \to b}} \frac{g(x, y+\varepsilon h_2) - g(x,y)}{\varepsilon}, \ h_2 \in Y. \end{aligned}$$

Definition B.4 *([13]) Let $f, g : K_1 \times K_2 \to \Re$ be two real-valued functions. The point $(a,b) \in K_1 \times K_2$ is called a **strong Nash stationary point** for the functions f, g if*

$$\begin{aligned} D_1^s f(a,b)(h_1) \geq 0 \ , \ \forall h_1 \in S_{K_1}(a) \\ D_2^s g(a,b)(h_2) \geq 0 \ , \ \forall h_2 \in S_{K_2}(b). \end{aligned}$$

Definition B.5 *([13]) Let $f \in \mathcal{L}_1(K_1 \times K_2)$,$g \in \mathcal{L}_2(K_1 \times K_2)$ be two real-valued function. The point $(a,b) \in K_1 \times K_2$ is called a **weak Nash stationary point** of the functions f, g if*

$$\begin{aligned} D_1^w f(a,b)(h_1) \geq 0 \ , \ \forall h_1 \in T_{K_1}(a) \\ D_2^w g(a,b)(h_2) \geq 0 \ , \ \forall h_2 \in T_{K_2}(b). \end{aligned}$$

The links between Nash equilibrium and Nash stationary points are developed in [13]. For our purpose, let us recall the following result.

Proposition B.6 *Let K_1, K_2 be nonempty compact convex sets and $f, g : K_1 \times K_2 \to \Re$ be functions such that $f \in \mathcal{L}_1(K_1 \times K_2), g \in \mathcal{L}_2(K_1 \times K_2)$ and*

$$\begin{array}{l} \forall y \in K_2 \text{ fixed, the function } x \in D_1 \longmapsto f(x,y) \text{ is convex} \\ \forall x \in K_1 \text{ fixed, the function } y \in D_2 \longmapsto g(x,y) \text{ is convex} \end{array}.$$

Then every weak Nash stationary point is a Nash equilibrium point.

Finally, we note some notions and results which belong to so called 'non-smooth analysis'. Consider V a real Banach space and U an open subset of V.

Definition B.7 *A function $f : U \to \Re$ is called a* ***locally Lipschitz function*** *if every $u \in U$ admits a neighborhood $N_u \subseteq U$ and a number $K = K(N_u) > 0$ such that*

$$|f(u_1) - f(u_2)| \leq K \|u_1 - u_2\|, \forall u_1, u_2 \in N_u.$$

We should remark that all convex continuous differentiable functions are locally Lipschitz functions.

The basic definition of the 'non-smooth analysis' is given below.

Definition B.8 *([9],[8],[10],[11],[12]) The* ***generalized directional derivative*** *of a locally Lipschitz function $f : U \to \Re$ at $u \in U$ in direction $v \in V$ is defined as*

$$f^0(u)(v) := \limsup_{\substack{w \to u \\ t \to 0^+}} \frac{f(w + tv) - f(w)}{t}.$$

We shall say that f is ***Clarke differentiable*** *at u if for every $v \in V$, the limit $f^0(u)(v)$ exists and it is finite.*

Recall the **classic directional derivative**

$$f'(u)(v) := \lim_{t \to 0^+} \frac{f(u + tv) - f(u)}{t}$$

when exists. It is obvious that, if $f'(u)(v)$ exists, then

$$f'(u)(v) \leq f^0(u)(v).$$

Some important results concerning the generalized directional derivative and the classic directional derivative are stated below.

Proposition B.9 *([1]) If $f : V \to \bar{\Re}$ is a strictly convex function, $u \in Domf$, $v \in V$, then exists $f'(u)(v)$ which satisfies*

$$f(u) - f(u - v) \leq f'(u)(v) \leq f(u + v) - f(u).$$

Proposition B.10 *([15])If $f : U \to \Re$ is continuous differentiable, then*

$$\langle \nabla f(u), v \rangle = f^0(u)(v) = f'(u)(v), \forall u \in U, \forall v \in V. \tag{1}$$

Proposition B.11 *([1]) If f is continuous at $u \in Domf$, then f is Clarke differentiable and*

$$f'(u)(v) = f^0(u)(v), \forall v \in V.$$

C The numerical algorithms

The following proposition will be employed in the numerical methods presented in this paper.

Proposition C.1 *Let $C_1 \subset \Re^{n_1}$, $C_2 \subset \Re^{n_2}$ be nonempty compact subsets, let $f, g : C_1 \times C_2 \to \Re$ be differentiable functions, let $x \in intC_1$, $y \in intC_2$, $u \in \Re^{n_1}$, $v \in \Re^{n_2}$ be arbitrary elements. If*

$$\left\langle u, \frac{\partial f}{\partial x}(x,y) \right\rangle < 0 \qquad \text{and} \qquad \left\langle v, \frac{\partial g}{\partial y}(x,y) \right\rangle < 0 \tag{2}$$

then there exists $a > 0$, $b > 0$ such that

$$\begin{array}{l} x + tu \in C_1 \ , \ f(x+tu, y) < f(x,y) \ , \ \forall t \in (0,a] \\ y + tv \in C_2 \ , \ g(x, y+tv) < g(x,y) \ , \ \forall t \in (0,b] \end{array}.$$

Example C.2 *If $\frac{\partial f}{\partial x}(x,y) \neq 0$ and $\frac{\partial g}{\partial y}(x,y) \neq 0$ then $u = -\frac{\partial f}{\partial x}(x,y)$ and $v = -\frac{\partial g}{\partial y}(x,y)$ are good candidates as descent directions satisfying (2).*

Example C.3 *If there exists $i \in \{1, 2, ..., n_1\}$ such that $\frac{\partial f}{\partial x_i}(x,y) \neq 0$ and $j \in \{1, 2, ..., n_2\}$ such that $\frac{\partial g}{\partial y_j}(x,y) \neq 0$ then $u = -sgn\frac{\partial f}{\partial x_i}(x,y)\, e^i$ and $v = -sgn\frac{\partial g}{\partial y_j}(x,y)\, e^j$ are good candidates as descent directions, as well.*

Recall that e^k is the vector with all components equal to zero, except the component on the position 'k', which is one.

We are now prepared to consider the following **problem (P):**

Find the Nash equilibrium for the two-person noncooperative game characterized by the following:

$K_1 \subset \Re, K_2 \subset \Re$ are nonempty compact convex subsets

$$K_1 = [m_1, M_1] \ , \ K_2 = [m_2, M_2]$$

$f, g : K_1 \times K_2 \to \Re$ are continuous differentiable functions ,with $f \in \mathcal{L}_1(K_1 \times K_2)$, $g \in \mathcal{L}_2(K_1 \times K_2)$ and

$$\begin{array}{l} \forall y \in K_2 \text{ fixed, the aplication } x \in D_1 \longmapsto f(x,y) \text{ is strictly convex} \\ \forall x \in K_1 \text{ fixed, the aplication } y \in D_2 \longmapsto g(x,y) \text{ is strictly convex} \end{array}.$$

According to the Theorem B.2, the problem (P) admits a Nash equilibrium point at least. Our goal is to elaborate numerical algorithms which supplies these points. Taking into account the Examples C.2 and C.3, we can elaborate two numerical algorithms, which generate a sequence $(x^p, y^p)_{p \in N^*}$ converging to a Nash equilibrium point.

<u>THE GRADIENT TYPE ALGORITHM:</u>

1. Choose $(x^1, y^1) \in (int[m_1, M_1]) \times (int[m_2, M_2])$, put $p = 1$;

2. Compute $\frac{\partial f}{\partial x}(x^p, y^p)$ and $\frac{\partial g}{\partial y}(x^p, y^p)$.

If $\left[\frac{\partial f}{\partial x}(x^p, y^p) = 0 \text{ and } \frac{\partial g}{\partial y}(x^p, y^p) = 0\right]$ put $x^q = x^p$, $y^q = y^p, \forall q > p$ and stop the algorithm;

3. If $\frac{\partial f}{\partial x}(x^p, y^p) \neq 0$ *then*
set $a_p := \left(1 - \frac{1}{2^p}\right) \sup\{\alpha > 0 : x^p - \alpha \frac{\partial f}{\partial x}(x^p, y^p) \in [m_1, M_1],$
$\forall \hat{\alpha} \in (0, \alpha], f\left(x^p - \hat{\alpha}\frac{\partial f}{\partial x}(x^p, y^p), y^p\right) < f(x^p, y^p)\}$
else set $a_p := 1$ *;*
4. If $\frac{\partial g}{\partial y}(x^p, y^p) \neq 0$ *then*
set $b_p := \left(1 - \frac{1}{2^p}\right) \sup\{\beta > 0 : y^p - \beta \frac{\partial g}{\partial y}(x^p, y^p) \in [m_2, M_2],$
$\forall \hat{\beta} \in (0, \beta], g\left(x^p, y^p - \hat{\beta}\frac{\partial g}{\partial y}(x^p, y^p)\right) < g(x^p, y^p)\}$
else set $b_p := 1$ *;*
5. Let $\left\{ \begin{array}{l} x^{p+1} := x^p - a_p \frac{\partial f}{\partial x}(x^p, y^p) \\ y^{p+1} := y^p - b_p \frac{\partial g}{\partial y}(x^p, y^p) \end{array} \right.$;
6. Increase p:=p+1 and go to the second step .

Consider now $(x^p, y^p)_{p \in N^*}$ the sequence obtained by the gradient type algorithm. Because the sequence $(x^p, y^p)_{p \in N^*}$ is bounded, Cesaro's Lemma implies that there exists a subsequence, denoted the same, converging to a certain point $(a, b) \in K_1 \times K_2$.

Theorem C.4 *The limit point* $(a, b) \in K_1 \times K_2$*, obtained by the gradient-type algorithm, is a Nash equilibrium point of the problem (P).*

A slight change in the gradient type method, based on the Example C.3, is given below.

<u>THE RELAXATION TYPE ALGORITHM:</u>

1. Choose $(x^1, y^1) \in (int\,[m_1, M_1]) \times (int\,[m_2, M_2])$ *, put* $p = 1$ *;*
2. Compute $\frac{\partial f}{\partial x}(x^p, y^p)$ *and* $\frac{\partial g}{\partial y}(x^p, y^p)$.
If $\left[\frac{\partial f}{\partial x}(x^p, y^p) = 0 \text{ and } \frac{\partial g}{\partial y}(x^p, y^p) = 0\right]$ *put* $x^q = x^p$, $y^q = y^p, \forall q > p$ *and stop the algorithm;*
3. If $\frac{\partial f}{\partial x}(x^p, y^p) \neq 0$ *then*
set $a_p := \left(1 - \frac{1}{2^p}\right) \sup\{\alpha > 0 : x^p - \alpha u \in [m_1, M_1],$
$\forall \hat{\alpha} \in (0, \alpha], f(x^p - \hat{\alpha} u, y^p) < f(x^p, y^p)\}$
where, $u = sgn \frac{\partial f}{\partial x}(x^p, y^p)$
else set $a_p := 1$ *;*
4. If $\frac{\partial g}{\partial y}(x^p, y^p) \neq 0$ *then*
set $b_p := \left(1 - \frac{1}{2^p}\right) \sup\{\beta > 0 : y^p - \beta v \in [m_2, M_2],$
$\forall \hat{\beta} \in (0, \beta], g\left(x^p, y^p - \hat{\beta} v\right) < g(x^p, y^p)\}$
where, $v = sgn \frac{\partial g}{\partial y}(x^p, y^p)$
else set $b_p := 1$ *;*
5. Let $\left\{ \begin{array}{l} x^{p+1} := x^p - a_p sgn \frac{\partial f}{\partial x}(x^p, y^p) \\ y^{p+1} := y^p - b_p sgn \frac{\partial g}{\partial y}(x^p, y^p) \end{array} \right.$;
6. Increase p:=p+1 and go to the second step .

Using the same argument as in the gradient-type method, we note that there exists a subsequence generated by the relaxation-type algorithm, which is convergent to a certain point $(a, b) \in K_1 \times K_2$.

Theorem C.5 *The limit point* $(a, b) \in K_1 \times K_2$ *obtained by the relaxation-type algorithm, is a Nash equilibrium point of the problem (P).*

Remark C.6 *The steps a_p, b_p from the gradient-type and relaxation-type algorithm exist, according to the Proposition C.1.*

The results stated above also hold for the n-player noncooperative games, which satisfy all the assumptions of the problem (P).

D Proofs

D.1 Proof of the basic proposition

We begin with the proof of the Proposition C.1.

We shall discuss only the conclusion about the function f. Since f is differentiable, then for every $(u', u'') \in \Re^{n_1} \times \Re^{n_2}$

$$\lim_{t\to 0} \frac{f(x+tu', y+tu'') - f(x,y)}{t} = \left\langle (u', u''), \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)\right\rangle . \tag{3}$$

If we choose $(u', u'') = (u, 0)$, the relation (3) becomes

$$\lim_{t\to 0} \frac{f(x+tu, y) - f(x,y)}{t} = \left\langle u, \frac{\partial f}{\partial x}(x,y)\right\rangle .$$

Then the limit definition, for $\varepsilon = -\left\langle u, \frac{\partial f}{\partial x}(x,y)\right\rangle > 0$, implies that

$$\exists a_1 > 0 \text{ s.t. } \left| \frac{f(x+tu, y) - f(x,y)}{t} - \left\langle u, \frac{\partial f}{\partial x}(x,y)\right\rangle \right| < -\left\langle u, \frac{\partial f}{\partial x}(x,y)\right\rangle ,$$

$$\forall t \in [-a_1, a_1] \setminus \{0\} .$$

Moreover, because $x \in intC_1$, it exists $a_2 > 0$ such that $x + tu \in C_1$, $\forall t \in [-a_2, a_2]$.

If we denote by $a = \min\{a_1, a_2\} > 0$, then for every $t \in (0, a]$, the following relations hold

$$\begin{aligned} x + tu &\in C_1 \\ f(x+tu, y) - f(x,y) &< 0. \end{aligned}$$

The second conclusion can be proven similarly. ■

D.2 Proof of the gradient-type theorem

The main result of this paper is Theorem C.4. Its proof is presented below.

After the gradient type algorithm is applied, a converging subsequence $(x^p, y^p)_{p\in N^*}$ is generated. Recall that (a, b) is its limit point. Now, we can be placed in one of the following situations: A, B, C, D, only.

Case A: The sequence $(x^p, y^p)_{p\in N^*}$ is obtained as the result of step 2.

Hence there exists $p_0 \in N$ such that $(x^p, y^p) = (a, b), \forall p \geq p_0$ and $\frac{\partial f}{\partial x}(a,b) = 0$, $\frac{\partial g}{\partial y}(a,b) = 0$.

Consider $\frac{\partial f}{\partial x}(a,b) = 0$.

We denote by $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)(x,y)$ the gradient of f at the point (x,y).

According to the Proposition B.10 we can write that

$$\langle \nabla f(x,y),(u,v)\rangle = f^0(x,y)(u,v) = \lim\sup_{\substack{\varepsilon\to 0^+\\ x'\to x\\ y'\to y}} \frac{f(x'+\varepsilon u, y'+\varepsilon v) - f(x',y')}{\varepsilon}.$$

If we take

$$\begin{aligned}(x,y) &= (a,b)\\ (u,v) &= (h,0)\ \text{(where } h\in\Re)\end{aligned}$$

in the relation above, we can note that

$$0 = \left\langle \frac{\partial f}{\partial x}(a,b), h\right\rangle + \left\langle \frac{\partial f}{\partial y}(a,b), 0\right\rangle = f^0(a,b)(h,0).$$

Moreover, for every $h\in\Re$ with $a-h\in K_1$, Proposition B.9 implies that

$$f(a,b) - f(a-h,b) \le f'(a,b)(h,0) \le f^0(a,b)(h,0) = 0$$

i.e.,

$$f(a,b) \le f(a-h,b), \forall h\in\Re \text{ with } a-h\in K_1.$$

So,

$$f(a,b) \le f(x,b), \forall x\in K_1. \tag{4}$$

Similarly, if we consider $\frac{\partial g}{\partial y}(a,b)=0$, we can deduce that

$$g(a,b) \le g(a,y), \forall y\in K_2. \tag{5}$$

The relations (4), (5) tell us that (a,b) is a Nash equilibrium point.
Case B: There is a subsequence, denoted similarly (x^p,y^p), s.t. $x^p\ne a, y^p\ne b, \forall p\in N^*$. In this situation we shall prove that (a,b) is a weak Nash stationary point, i.e.,

$$D_1^w f(a,b)(h_1) \ge 0\ ,\ \forall h_1\in T_{K_1}(a)$$

$$D_2^w g(a,b)(h_2) \ge 0\ ,\ \forall h_2\in T_{K_2}(b).$$

We prove that

$$\text{for every } h_1\in T_{K_1}(a), D_1^w f(a,b)(h_1) \ge 0. \tag{6}$$

Let us assume that there exists $h\in T_{K_1}(a)$ such that

$$l = \lim\sup_{\substack{\varepsilon\to 0^+\\ x\to a, y\to b (y\in K_2)}} \frac{f(x+\varepsilon h, y) - f(x,y)}{\varepsilon} < 0. \tag{7}$$

Because $l<0$, there exists $l'>0$ such that $l+l'<0$.
Moreover, inequality (7) implies that

$$\exists\delta_1>0: \frac{f(x+\varepsilon h,y)-f(x,y)}{\varepsilon} < l+l' < 0$$

for $\forall 0<\varepsilon<\delta_1$, for $\forall x\ne a, |x-a|<\delta_1$, for $\forall y\in K_2, y\ne b, |y-b|<\delta_1$.

The Mean Value Theorem implies further that

$$\exists \xi_\varepsilon \in (x, x+\varepsilon h) : \frac{\left\langle \frac{\partial f}{\partial x}(\xi_\varepsilon, y), \varepsilon h \right\rangle}{\varepsilon} < l + l' < 0$$

i.e.,

$$\exists \xi_\varepsilon \in (x, x+\varepsilon h) : \left\langle \frac{\partial f}{\partial x}(\xi_\varepsilon, y), h \right\rangle < l + l' < 0 \tag{8}$$

for $\forall 0 < \varepsilon < \delta_1$, for $\forall x \neq a, |x-a| < \delta_1$, for $\forall y \in K_2, y \neq b, |y-b| < \delta_1$.
Relation (8) contains two relevant information:

1. $h \neq 0$.
2. $\frac{\partial f}{\partial x}(x,y) \neq 0, \forall x \neq a, |x-a| < \delta_1, \forall y \in K_2, y \neq b, |y-b| < \delta_1$.

   Otherwise, if $\frac{\partial f}{\partial x}(x,y) = 0$, because of the continuity of the derivative, there exists $\delta' > 0$ s.t. $\left|\frac{\partial f}{\partial x}(x', y)\right| < \frac{|l+l'|}{|h|}$, for $\forall x'$, $|x'-x| < \delta'$. This implies

$$\left|\left\langle \frac{\partial f}{\partial x}(x', y), h \right\rangle\right| \leq \left|\frac{\partial f}{\partial x}(x', y)\right| \cdot |h| < |l + l'| \tag{9}$$

   Consider $\varepsilon \in (0, \delta_1), |\varepsilon h| < \delta'$. Then, by (8) we can remark that

$$\left|\left\langle \frac{\partial f}{\partial x}(\xi_\varepsilon, y), h \right\rangle\right| \geq |l + l'|, \text{ where } |\xi_\varepsilon - x| < |\varepsilon h| < \delta'.$$

   The last evaluation is in contradiction with the relation (9).

Thus, we can note that

$$\frac{\partial f}{\partial x}(x,y) \neq 0, \forall x \neq a, |x-a| < \delta_1, \forall y \in K_2, y \neq b, |y-b| < \delta_1 \tag{10}$$

Hence, by case A hypothesis, by relations (8) and (10), there exists a subsequence, denoted similarly $(x^p, y^p)_{p\in N^*}$, and $p_0 \in N^*$ with the following properties:

$$x^p \neq a, |x^p - a| < \delta_1, y^p \in K_2, y^p \neq b, |y^p - b| < \delta_1, \forall p \geq p_0$$

$$\frac{\partial f}{\partial x}(x^p, y^p) = -\lambda(x^p, y^p) \cdot h \tag{11}$$

where,

$$\lambda(x^p, y^p) > 0, \forall p \geq p_0.$$

The following three subcases are possible: $a \in intK_1, a = m_1, a = M_1$.

If $a \in intK_1$ there exists $\delta_2 > 0$ s.t. $(a-\delta_2, a+\delta_2) \subset intK_1$.

Let $\delta_3 = \min\left\{\delta_1, \delta_2, \frac{\delta_2}{1+|h|}\right\} > 0$. Corresponding to $\delta_3 > 0$, there exists $p_1 \in N^*$ s.t.:

$$|x^p - x^q| < \frac{\delta_3}{2}, \forall p, q \geq p_1 \tag{12}$$

$$|x^p - a| < \delta_3, \forall p \geq p_1$$
$$|y^p - b| < \delta_3, \forall p \geq p_1 \, (y^p \in K_2).$$

Denote by $p_2 = \max\{p_0, p_1\}$. Then, taking into account the relation (12), we can note that for every $p \geq p_2$, and $0 < \varepsilon < \delta_3$ follow

$$|x^p + \varepsilon h - a| \leq |x^p - a| + \varepsilon |h| < \delta_3 + \delta_3 |h| \leq \delta_2.$$

So,

$$x^p + \varepsilon h \in (a - \delta_2, a + \delta_2) \subseteq intK_1$$
$$x^{p+1} \in (a - \delta_3, a + \delta_3) \subseteq (a - \delta_2, a + \delta_2) \subseteq intK_1.$$

Recall from gradient type algorithm, that x^{p+1} has the form $x^{p+1} = x^p - a_p \frac{\partial f}{\partial x}(x^p, y^p)$. Looking at the function values, one can notice that

$$f(x^p + \varepsilon h, y^p) - f(x^p, y^p) < 0 \text{ (from (7))}$$
$$f\left(x^p - a_p \frac{\partial f}{\partial x}(x^p, y^p), y^p\right) - f(x^p, y^p) < 0 \, \text{(from algorithm)}$$

The relations above, relation (11) and the gradient-type algorithm imply

$$a_p \lambda(x^p, y^p) \geq \varepsilon \left(1 - \frac{1}{2^p}\right), \forall \varepsilon \text{ with } 0 < \varepsilon < \delta_3$$

i.e.,

$$a_p \lambda(x^p, y^p) \geq \delta_3 \left(1 - \frac{1}{2^p}\right), \forall p \geq p_2. \tag{13}$$

Finally, by relations (11) and (13), we can conclude that

$$\frac{\delta_3}{2} > \left|x^{p+n} - x^p\right|$$
$$= \left|a_p \lambda(x^p, y^p) h + a_{p+1} \lambda\left(x^{p+1}, y^{p+1}\right) h + \dots + a_{p+n} \lambda\left(x^{p+n}, y^{p+n}\right) h\right|$$
$$\geq |h| (n+1) \delta_3 \left(1 - \frac{1}{2^p}\right) \geq |h| (n+1) \frac{\delta_3}{2}, \forall p \geq p_3$$

i.e.,

$$(n+1)|h| < 1, \ \forall n \in N^*.$$

In other words, we obtained a contradiction.

If $a = m_1$, then taking into account that $h > 0$, we get

$$x^{p+1} = x^p + a_p \lambda(x^p, y^p) h > x^p > a = m_1, \forall p \geq p_0$$

and we obtained a contradiction with the fact that $x^p \to a$.

If $a = M_1$, then similarly we get

$$x^{p+1} = x^p + a_p \lambda (x^p, y^p) h < x^p < a = M_1 \ ,\forall p \geq p_0$$

and we obtained a contradiction with the same $x^p \to a$.

All the contradictions obtained earlier show us that

$$\forall h_1 \in T_{K_1}(a), D_1^w f(a,b)(h_1) \geq 0.$$

To show that for every $h_2 \in T_{K_2}(b)$, $D_2^w g(a,b)(h_2) \geq 0$, we proceed in the similar way.

As a conclusion, by Proposition B.6, one can see that (a, b) is a Nash equilibrium point.

Case C: There exists a subsequence, denoted similarly $(x^p, y^p), p_0 \in N^*$ s.t. $x^p \neq a, \forall p \in N^*, y^p = b, \forall p \geq p_0$.

We prove that (a, b) is a weak Nash stationary point.

Our goal is to prove that for every $h_1 \in T_{K_1}(a)$, $D_1^w f(a,b)(h_1) \geq 0$. Again this will be proven by contradiction.

First, we have to remark that $b \in intK_2$, according to the gradient type algorithm.

Now, we assume that $\exists h \in T_{K_1}(a)$:

$$l = \lim \sup_{\substack{\varepsilon \to 0^+ \\ x \to a, y \to b (y \in K_2)}} \frac{f(x+\varepsilon h, y) - f(x,y)}{\varepsilon} < 0.$$

We denote by $f_b : K_1 \to \Re$, the following continuous differentiable function given by $f_b(x) := f(x,b)$.

Then, using Propositions B.10 and B.11, we can deduce the following estimate, for all $h \in \Re$:

$$\lim \sup_{\substack{\varepsilon \to 0^+ \\ x \to a}} \frac{f(x+\varepsilon h, b) - f(x,b)}{\varepsilon} = f_b^0(a)(h) = f_b'(a)(h)$$

$$= \lim_{\varepsilon \to 0^+} \frac{f_b(a+\varepsilon h) - f_b(a)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{f(a+\varepsilon h, b) - f(a,b)}{\varepsilon}$$

$$= f'(a,b)(h,0) = f^0(a,b)(h,0) = \lim \sup_{\substack{\varepsilon \to 0^+ \\ x \to a, y \to b}} \frac{f(x+\varepsilon h, y) - f(x,y)}{\varepsilon}$$

$$= \lim \sup_{\substack{\varepsilon \to 0^+ \\ x \to a, y \to b (y \in K_2)}} \frac{f(x+\varepsilon h, y) - f(x,y)}{\varepsilon} = l < 0.$$

Hence,

$$\lim \sup_{\substack{\varepsilon \to 0^+ \\ x \to a}} \frac{f(x+\varepsilon h, b) - f(x,b)}{\varepsilon} < 0. \tag{14}$$

Following the same argument as in the case B, we obtain a contradiction. So $\forall h_1 \in T_{K_1}(a)$, $D_1^w f(a,b)(h_1) \geq 0$.

The next step is to prove that for every $h_2 \in T_{K_2}(b)$, $D_2^w g(a,b)(h_2) \geq 0$.

Since there exists $p_0 \in N^*$ s.t. $y^p = b, \forall p \geq p_0$ it follows that $\frac{\partial g}{\partial y}(x^p, y^p) = \frac{\partial g}{\partial y}(x^p, b) = 0, \forall p \geq p_0$.

Let $h_2 \in S_{K_2}(b)$ be chosen arbitrarily, let $\varepsilon > 0$ be s.t. $b + \varepsilon h_2 \in K_2$ (it is possible because $b \in int K_2$). By [13], we can write that

$$D_2^w g(a,b)(h_2) \geq D_2^s g(a,b)(h_2), h_2 \in S_{K_2}(b). \tag{15}$$

Having the last inequality in mind, we shall prove that $D_2^s g(a,b)(h_2) \geq 0$.
Denote by $g_p : K_2 \to \Re$, the function defined as $g_p(y) = g(x^p, y)$, where $p \geq p_0$ is fixed. According to the Propositions B.9 and B.10, we can infer that

$$0 = \langle \nabla g(x^p, b), (0, \varepsilon h_2) \rangle = \left\langle \frac{\partial g}{\partial y}(x^p, b), \varepsilon h_2 \right\rangle = \left\langle \frac{\partial g_p}{\partial y}(b), \varepsilon h_2 \right\rangle$$

$$= g_p'(b)(\varepsilon h_2) \leq g_p(b + \varepsilon h_2) - g_p(b) = g(x^p, b + \varepsilon h_2) - g(x^p, b).$$

Hence, for every $p \geq p_0$, we got $g(x^p, b + \varepsilon h_2) - g(x^p, b) \geq 0$. Letting $p \to \infty$, we obtain

$$g(a, b + \varepsilon h_2) - g(a,b) \geq 0.$$

So

$$D_2^s g(a,b)(h_2) := \lim \inf_{\substack{\varepsilon \to 0^+ \\ b + \varepsilon h_2 \in K_2}} \frac{g(a, b + \varepsilon h_2) - g(a,b)}{\varepsilon} \geq 0. \tag{16}$$

Because $h_2 \in S_{K_2}(b)$ was chosen arbitrarily, relations (15) and (16) imply

$$D_2^w g(a,b)(h_2) \geq 0, \forall h_2 \in S_{K_2}(b).$$

Because of the continuity with respect to h_2, one can obtain that

$$D_2^w g(a,b)(h_2) \geq 0, \forall h_2 \in T_{K_2}(b).$$

Again, according to the Proposition B.6, we have that (a,b) is a Nash equilibrium point.

Case D: There exists a subsequence, denoted similarly $(x^p, y^p), p_0 \in N^*$ s. t. $x^p = a, \forall p \geq p_0, y^p \neq b, \forall p \in N^*$.

The prove is similarly with that of case C, so we can conclude that (a,b) is a Nash equilibrium point.

Since all the possible situations were analyzed, we can say that (a,b) is a Nash equilibrium point for the problem (P). ■

D.3 Proof of the relaxation-type theorem

The prove of the Theorem C.5 follows the same idea as the proof of the Theorem C.4.

E Illustrative example

Let us consider the following 2-player noncooperative game example.
Consider the game given by

$$f : [0,2] \times [1,3] \to \Re,\ f(x,y) = 2x^2 - 2xy + 5y^2 - 6x - 6y$$

$$g : [0,2] \times [1,3] \to \Re,\ g(x,y) = x^2 + xy + y^2 - 3x - 6y.$$

These functions satisfy the assumptions of the problem (P).

First, let us compute the Nash equilibrium point exactly. Denote by $(a, b) \in [0,2] \times [1,3]$ the Nash equilibrium. So,

$$\begin{array}{l} f(a,b) \leq f(x,b), \forall x \in [0,2] \\ g(a,b) \leq g(a,y), \forall y \in [1,3] \end{array}$$

i.e., $\left\{ \begin{array}{l} 2a^2 - 2ab + 5b^2 - 6a - 6b \leq 2x^2 - 2xb + 5b^2 - 6x - 6b, \forall x \in [0,2] \\ a^2 + ab + b^2 - 3a - 6b \leq a^2 + ay + y^2 - 3a - 6y, \forall y \in [1,3] \end{array} \right.$

$$\Rightarrow \left\{ \begin{array}{l} 2(a-x)(a+x) - 2b(a-x) - 6(a-x) \leq 0, \forall x \in [0,2] \\ a(b-y) + (b-y)(b+y) - 6(b-y) \leq 0, \forall y \in [1,3] \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} (a-x)[2(a+x) - 2b - 6] \leq 0, \forall x \in [0,2] \\ (b-y)[(b+y) + a - 6] \leq 0, \forall y \in [1,3] \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} (a-x)[x + (a-b-3)] \leq 0, \forall x \in [0,2] \\ (b-y)[y + (a+b-6)] \leq 0, \forall y \in [1,3] \end{array} \right. . \qquad (17)$$

Notice that the following relation is satisfied

$$x + (a-b-3) \leq 0, \forall x \in [0,2], \forall a \in [0,2], \forall b \in [1,3].$$

Hence, we only have to impose the condition

$$a - x \geq 0, \forall x \in [0,2] \text{ which gives } a = 2.$$

Under this circumstance, the second inequality of the system (17) becomes

$$(b-y)[y + (b-4)] \leq 0, \forall y \in [1,3].$$

It is obvious that the condition we have to impose here is $b = 4 - b$. This means $b = 2$.

As a conclusion, the Nash equilibrium point is $N(2,2)$.

Now, let us apply the numerical algorithms presented in the paper. The evolutions of the game under the gradient-type and the relaxation-type algorithm are depicted in Figure 1 and Figure 2, respectively. The starting point was chosen as $(0.2; 1.2)$. Moreover, for the sake of simplicity, the original gradient-type and relaxation-type algorithm were modified a little, in order to avoid solving maximization problems. The stepsizes a_p, b_p were considered to be the biggest stepsizes which satisfy both conditions: the new generated point is inside the interval and the function value is less at this new point than the value at the old point.

For this particular example, it is observed that the relaxation-type algorithm reaches the Nash equilibrium in fewer steps than gradient-type algorithm.

Finally, let us solve the same problem with the algorithms proposed by S. Li and T. Basar ([14]) and S. Uryas'ev and R. Rubinstein ([18]). Figure 3 and 4 presents the results, respectively. We need to emphasize that, in the last two numerical algorithms, the corresponding minimization/maximization problems were solved exactly.

F Conclusions

We have developed two numerical methods for computation of Nash equilibria. As the numerical example shows, the results are quite satisfactory. Plus, we can underline some remarks. First, the gradient-type and relaxation-type algorithms do not require much information, only the expressions of the functions and their gradients. Second, because the stepsize

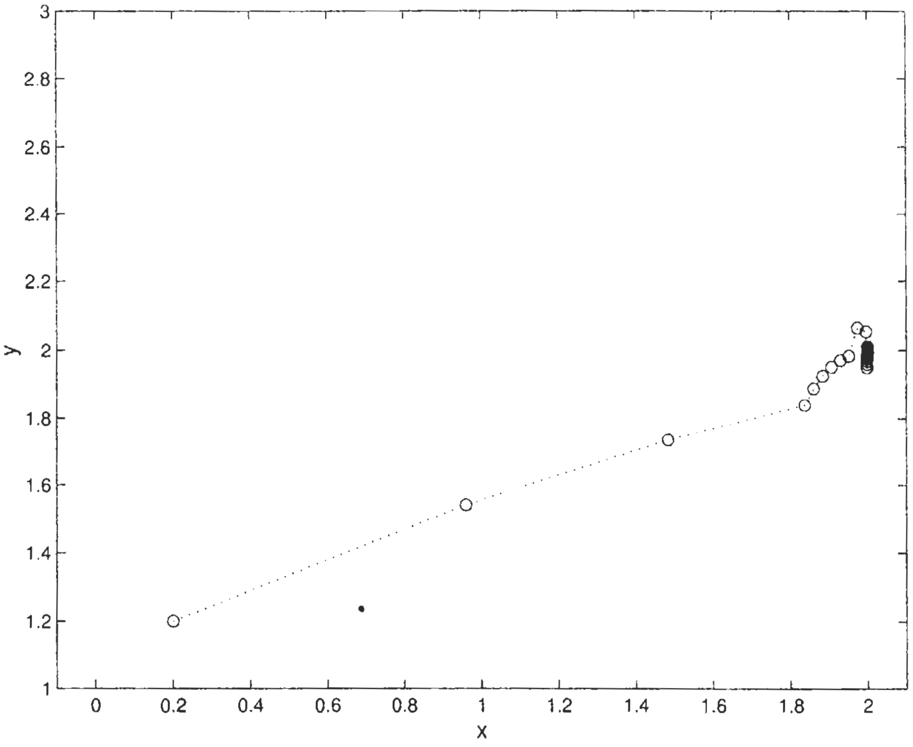

Figure 1: **Gradient-type algorithm**

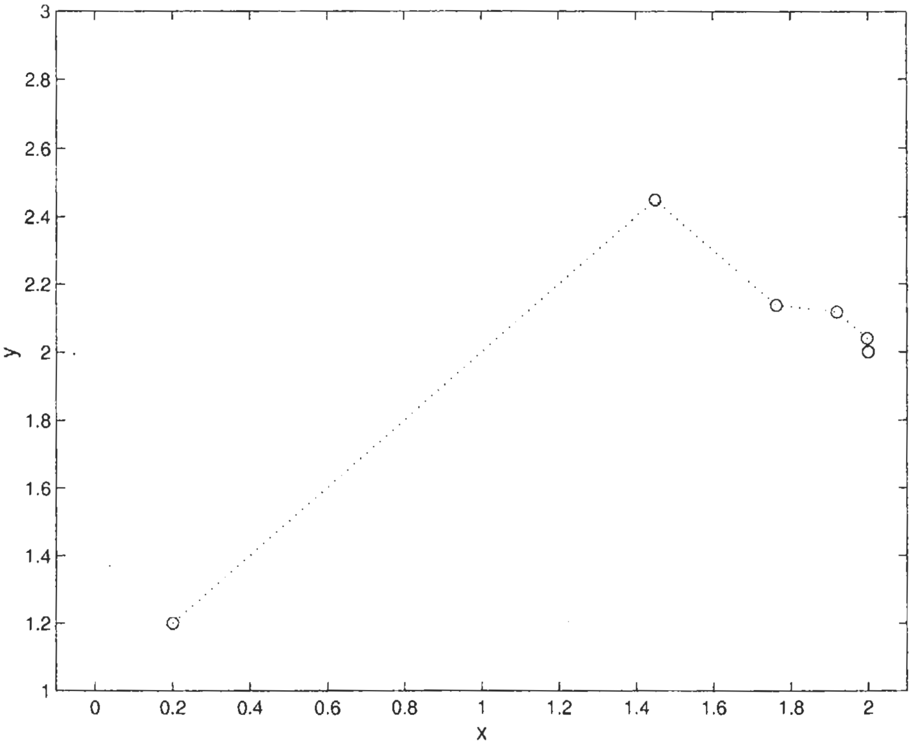

Figure 2: **Relaxation-type algorithm**

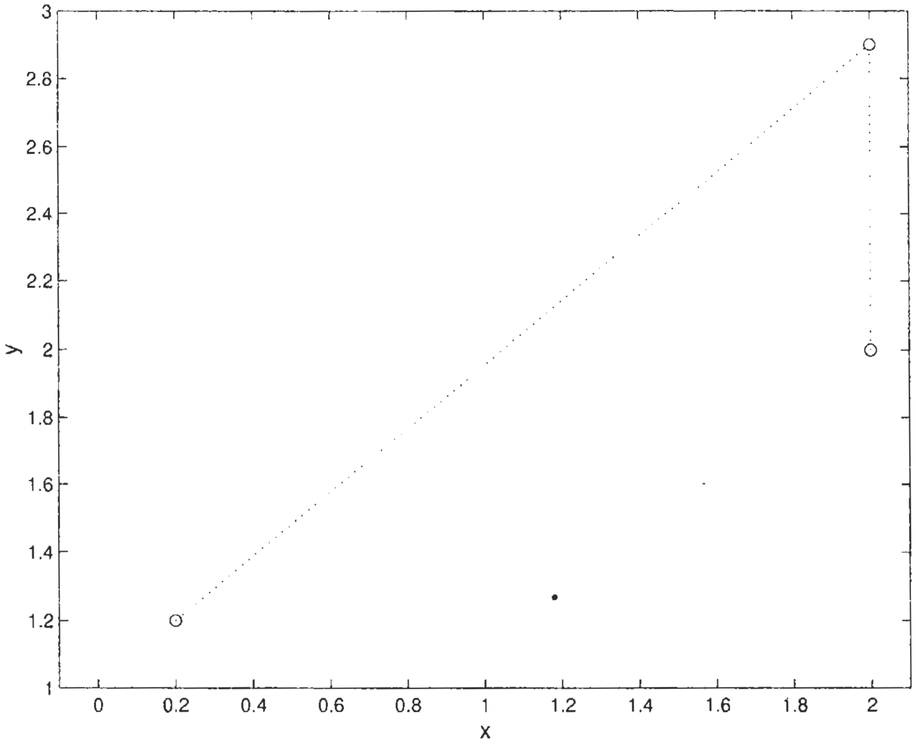

Figure 3: **Li and Basar algorithm**

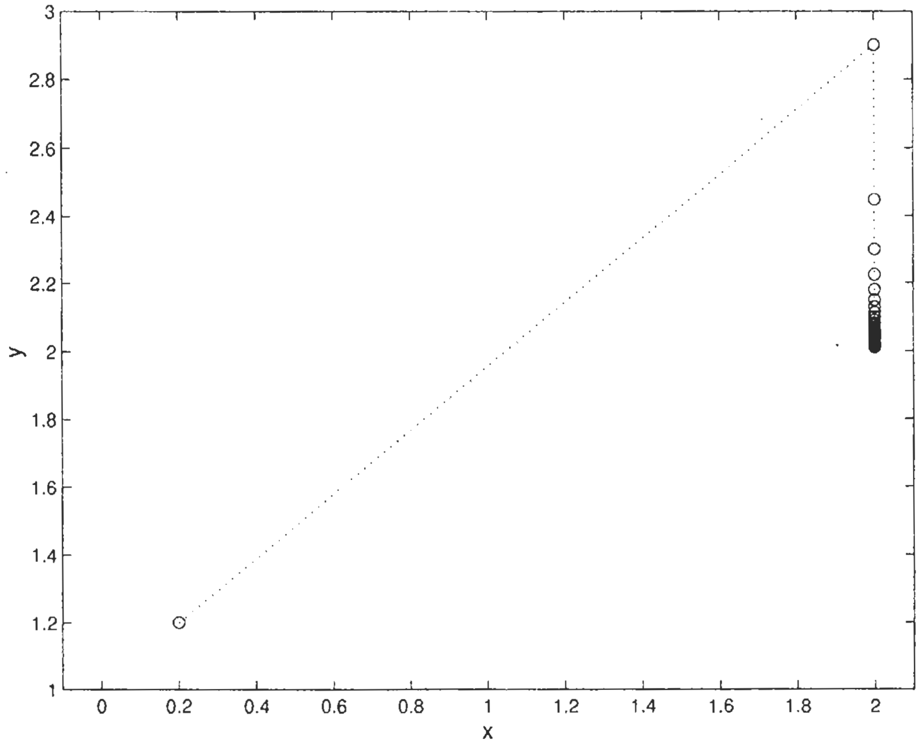

Figure 4: **Uryas'ev and Rubinstein algorithm**

search may be simplified, as we did in our example, the implementation of these numerical methods becomes trivial. We may conclude that the two numerical methods presented here can be successfully applied to all noncooperative games.

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