Section 5.3: Mean and Standard Deviation for a Discrete Random Variable

After constructing a probability model, it is easy to determine the mean and standard deviation for the given experiment.

Definition 1 *The* ***expected value*** *(or* ***mean****) of a discrete probability distribution is given by*

$$E(X) = \sum_{x \in X} x * p(x).$$

Definition 2 *The* ***standard deviation*** *of a discrete probability distribution is given by*

$$\sigma = \sqrt{\sum_{x \in X} (x - E(X))^2 * p(x)}.$$

Example 3 *Recall the experiment of rolling a pair of dice and summing the faces. The random variable X assigns to each roll its sum. The following table is our previously constructed probability model.*

Sum	2	3	4	5	6	7	8	9	10	11	12
Prob	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

What is the expected value of the sum of a pair of dice?

$$\begin{aligned}
E(X) &= \sum_{x \in X} x * p(x) \\
&= 2 * \frac{1}{36} + 3 * \frac{2}{36} + 4 * \frac{3}{36} + 5 * \frac{4}{36} + 6 * \frac{5}{36} + 7 * \frac{6}{36} \\
&\quad + 8 * \frac{5}{36} + 9 * \frac{4}{36} + 10 * \frac{3}{36} + 11 * \frac{2}{36} + 12 * \frac{1}{36} \\
&= \frac{2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12}{36} \\
&= 7.
\end{aligned}$$

What is the standard deviation of the sum of a pair of dice?

Sum	2	3	4	5	6	7	8	9	10	11	12
$(x - E(X))^2$	25	16	9	4	1	0	1	4	9	16	25
Prob	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$
\begin{aligned}
\sigma^2 &= \sum_{x\in X} (x-E(X))^2 * p(x) \\
&= 25 * \frac{1}{36} + 16 * \frac{2}{36} + 9 * \frac{3}{36} + 4 * \frac{4}{36} + 1 * \frac{5}{36} + 0 * \frac{6}{36} \\
&\quad +1 * \frac{5}{36} + 4 * \frac{4}{36} + 9 * \frac{3}{36} + 16 * \frac{2}{36} + 25 * \frac{1}{36} \\
&= \frac{35}{6}.
\end{aligned}
$$

Thus, $\sigma = \sqrt{\frac{35}{6}} = 2.4152$.

Example 4 *What is the probability that the sum of two dice falls within one standard deviation of the mean?*
One standard deviation from the mean is $(7 \pm 2.4152) = (4.5848, 9.4152)$*. So we could see a sum of 5, 6, 7, 8, or 9. This occurs with probability*

$$
p = \frac{4}{36} + \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36} = \frac{2}{3} = 0.66667.
$$

Problem 5 *Determine the expected value and standard deviation for the following probability distribution W.*

w	1	3	5	9
$p(w)$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

Exercise 6 *What is the probability that outcome falls more than one standard deviation away from the mean?*

Exercise 7 *A \$2 lottery ticket offers four chances to win different amounts of money as indicated by the following probability distribution model. Determine the expected value of buying a single ticket. Do you wish to play this game?*

Prize	\$0	\$0.5	\$1	\$5	\$50
Prob	$\frac{71}{100}$	$\frac{15}{100}$	$\frac{10}{100}$	$\frac{3}{100}$	$\frac{1}{100}$
Value	-2	-1.5	-1	3	48

So,

$$\begin{aligned} E(X) &= \sum_{x \in X} x * p(x) \\ &= -2 * \frac{71}{100} - 1.5 * \frac{15}{100} - 1 * \frac{10}{100} + 3 * \frac{3}{100} + 48 * \frac{1}{100} \\ &= -1.175. \end{aligned}$$

I personally would not want to play this game with a negative expected value and such a small prize.

Exercise 8 *What is the expected value of buying 50 such lottery tickets?*
$-1.175 * 50 = -58.75$.

Problem 9 *You pay \$1 to play a game. The game consists of rolling a pair of dice. If you observe a sum of 7 or 11 you receive \$4. If not, you receive nothing. Compute the expected value and standard deviation for this game?*

1 Exercises

Kokoska 3rd edition Section 5.3: 5.52-5.55, 5.62, 5.63, 5.67, 5.68, 5.75