Section 6.1: Probability Distributions for a Continuous Random Variable

A continuous random variable contains an uncountably infinite number of different values making it impossible to list each one. A different approach for a probability model is needed. Consider an unskilled player throwing darts at a board. Different regions of different sizes exist on the board. The likelihood of an unskilled player hitting the center white ring is not the same as hitting anywhere in the 20 region. How might we approach the assignment of probabilities for the dartboard?

Let's assume a dart always hits in the scoring area (which is a big assumption) and all points of contact on the board equally likely. We now let the relative area of a region on the board serve as the probability that a dart lands in said region.

Definition 1 *A probability distribution for a continuous random variable X is given by a* ***probability density function (pdf)*** *$f(x)$. The probability that X takes a value in the interval $[a,b]$ is the area under the $f(x)$ from a to b. Every pdf satisfies two properties.*

1. *The total area under the curve defined by $f(x)$ is 1;*
2. *and $f(x) \geq 0$ for all x.*

Example 2 *Is $f(x) = x$ on the domain $[0,1]$ a pdf? No, because the area in the triangle generated by the curve is $\frac{1*1}{2} = \frac{1}{2} < 1$.*

Example 3 *Is $f(x) = x$ on the domain $[0,3]$ a pdf? No, because the area in the triangle generated by the curve is $\frac{3*3}{2} = \frac{9}{2} > 1$*

Example 4 *Is $f(x) = \frac{x}{18}$ on the domain $[0,6]$ a pdf? Yes, since the area in the triangle generated by the curve is $\frac{6*\frac{6}{18}}{2} = 1$ and $f(x) \geq 0$ for all $x \in [0,6]$. .*

Problem 5 *Is $f(x) = x$ on the domain $[-1,2]$ a pdf?*

Problem 6 *Is* $f(x) = x$ *on the domain* $[2, 3]$ *a pdf?*

Example 7 *For the pdf* $f(x) = \frac{x}{18}$ *with domain* $[0, 6]$ *, what is the probability that an observations falls in the interval* $[1, 3]$*? The area in* $[0, 3]$ *is* $\frac{3*\frac{3}{18}}{2} = \frac{1}{4}$*. The area in* $[0, 1]$ *is* $\frac{1*\frac{1}{18}}{2} = \frac{1}{36}$*. The* $P(x \in [1, 3]) = \frac{1}{4} - \frac{1}{36} = \frac{2}{9}$*.*

Example 8 *For the pdf* $f(x) = \frac{x}{18}$ *with domain* $[0, 6]$ *, what is the probability that* $x = 5$*? Note that* $P(x = 5) = 0$ *since the line at* $x = 5$ *is one dimensional and has no area.*

Problem 9 *For the pdf* $f(x) = \frac{x}{18}$ *with domain* $[0, 6]$ *, what is the probability that an observations falls in the interval* $[2, 5]$*?*

Problem 10 *For the pdf* $f(x) = \frac{x}{18}$ *with domain* $[0, 6]$ *, what is the probability that an observations falls in the interval* $[5, 6]$*?*

Definition 11 *The* ***uniform distribution*** *is distributed uniformly between two points a and b. In a uniform distribution*

1. *the pdf* $f(x) = \frac{1}{b-a}$ *on the domain* $[a, b]$*;*
2. $\mu = \frac{a+b}{2}$*;*
3. *and* $\sigma = \frac{b-a}{\sqrt{12}}$*.*

Note that $f(x) = \frac{1}{b-a}$ is a horizontal line. The area under $f(x) = \frac{1}{b-a}$ on any interval forms a rectangle.

Example 12 *Consider the uniform distribution the interval* $[6, 10]$*.*

1. *Find the mean and standard deviation for this distribution.*
   $\mu = \frac{a+b}{2} = \frac{6+10}{2} = 8$ *and* $\sigma = \frac{b-a}{\sqrt{12}} = \frac{10-6}{\sqrt{12}} = 1.1547$
2. *Find the pdf for this distribution.*
   $f(x) = \frac{1}{b-a} = \frac{1}{10-6} = \frac{1}{4}$
3. *Compute* $P(x \in [7, 8.5])$*.*
   The area of the rectangle from 7 to 8.5 under the curve $f(x)$ *is* $1.5 * \frac{1}{4} = 0.375$*. So the probability that a randomly selected observation falls into the interval* $[7, 8.5]$ *is 0.375.*

4. *Compute* $P(x > \mu + 2\sigma)$.
   $\mu + 2\sigma = 8 + 2 * 1.1547 = 10.309$. *The probability that* $P(x > 10)$ *on the domain* $[6, 10]$ *is 0.*

Problem 13 *Consider the uniform distribution the interval* $[0, 10]$.

1. *Find the mean and standard deviation for this distribution.*

2. *Find the pdf for this distribution.*

3. *Compute* $P(x \in [4.7, 8.3])$.

4. *Compute* $P(x < \mu + \sigma)$.

1 Exercises

1. Let $f(x) = \frac{x}{c}$ on the domain $[0, 10]$. Find the constant c that makes $f(x)$ a pdf.
2. Let $f(x) = \frac{x}{c}$ on the domain $[0, 5]$. Find the constant c that makes $f(x)$ a pdf.
3. Kokoska 3rd edition Section 6.1: 6.10-6.15, 6.17, 6.19, 6.22, 6.25