Section 2.1: Probability

1 Pop Quiz!

1. Dr. DeMaio's favorite sport on ESPN 8 is:
   a) Curling
   b) Dodgeball
   c) Lawn Mower Racing
   d) Caber Tossing
   e) Squirrel Water-Skiing

2. On college intramural teams, Dr. DeMaio's jersey number was always:
   a) π
   b) 13
   c) $\sqrt{-1}$
   d) $\lim_{x \to 0} \frac{1}{x}$

2 Formal Probability

For the pop quiz questions above random guessing had to be employed. The probability of a correct answer was one out of the number of responses offered. This concept of counting and dividing is the heart of computing probabilities. What is the probability of a correct answer on the pop quiz? For #1, $p = \frac{1}{5} = .2 = 20\%$. For #2, $p = \frac{1}{4} = .25 = 25\%$.

Example 1 *Dr. DeMaio's morning MATH 1107 class contains 12 freshmen, 23 sophomores, 5 juniors and 11 seniors. If one student is selected at random, what is the probability that they are a senior? There are* $12 + 23 + 5 + 11 = 51$ *students in the class. Thus, the probability that a senior is selected is* $p = \frac{11}{51} = 0.215\,69 = 21.6\%$.

Theorem 1 *The **Law of Large Numbers** (LLN) says that the long-run relative frequency of repeated independent events gets closer and closer to the true relative frequency as the number of trials increases.*

For example, consider flipping a fair coin many, many times. The overall percentage of heads should settle down to about 50% as the number of outcomes increases.

The common (mis)understanding of the LLN is that random phenomena are supposed to compensate for whatever happened in the past. This is just not true. For example, when flipping a fair coin, if heads comes up on each of the first 10 flips, the probability of a tail on the next flip is still $p = \frac{1}{2}$. The coin does not remember what it did in the past. The coin does not feel bad that a tail has not been given a turn recently.

Thanks to the LLN, we know that relative frequencies settle down in the long run, so we can officially give the name probability to that value.

To compute a probability without running the experiment countless times to determine the true relative frequency of an event we consider all the equally likely possible outcomes of an experiment. It's equally likely to get any one of six outcomes from the roll of a fair die. It's equally likely to get heads or tails from the toss of a fair coin. However, keep in mind that events are not always equally likely. A skilled basketball player has a better than 50-50 chance of making a free throw. When rolling a pair of dice, a sum of seven and a sum of twelve are not equally likely events.

Definition 1 *A **random phenomenon** is a situation in which we know what outcomes could happen, but we don't know which particular outcome did or will happen.*

For any random phenomenon, each attempt, or trial, generates an outcome. Something happens on each trial, and we call whatever happens the outcome. These outcomes are individual possibilities, like the number we see on top when we roll a die. Sometimes we are interested in a combination of outcomes (e.g., a die is rolled and comes up even). A combination of outcomes is called an **event**.

Example 2 *An experiment consists of randomly picking a card from a standard deck of playing cards (no jokers). Some possible outcomes of this experiment are listed below.*

A - The 8 of clubs is selected.
B - A red card is selected.
C - The Jack of hearts is not selected.

Example 3 *An experiment consists of watching ten summer blockbuster movies and counting the number of Coca-Cola product placements.*

Remark 2 *On many occasions we will want to perform mathematical operations on events.*

Definition 2 *The event **A complement**, denoted $\bar{A}$ (A' or A^c), is the event that A does not occur.*

Definition 3 *The event **A union B**, denoted $A \cup B$, is the event that A or B (or both) occur.*

Definition 4 *The event **A intersect B**, denoted $A \cap B$, is the event that A and B both occur.*

Problem 1 *Using a Venn diagram, shade each of the above operations.*

Example 4 *For the experiment that consists of randomly picking a card from a standard deck of playing cards (no jokers) consider the following outcomes.*

A - The 8 of clubs is selected.
B - A red card is selected.
C - The Jack of hearts is not selected.

Problem 2 *Describe each of the following:*

1. $\bar{A}$
2. $\bar{B}$
3. $\bar{C}$
4. $A \cup B$
5. $A \cap B$
6. $A \cap C$
7. $B \cup C$
8. $B \cap C$
9. $\overline{B \cap C}$
10. $\overline{B} \cup \overline{C}$

Problem 3 *For each of the above events, compute its probability.*

We also want the events to be **disjoint** (or **mutually exclusive**). Two events are disjoint if they cannot occur at the same time. Mathematically speaking, A and B are disjoint if and only if $A \cap B = \emptyset$.

Example 5 *An experiment consists of randomly picking a card from a standard deck of playing cards (no jokers). Which pairs of the following events are disjoint?*

A - The 8 of clubs is selected.
B - A red card is selected.
C - The Jack of hearts is not selected.

Definition 5 *Let an experiment consist of a collection of s disjoint and equally likely events. This collection is called the **sample space**. Furthermore suppose exactly n of the events result in event A. Then the probability that event A will occur is $P(A) = \frac{n}{s}$. Furthermore, the probability of the set of all possible outcomes of an experiment must be $P(S) = 1$.*

Probabilities must be between 0 and 1, inclusive. A probability of 0 indicates impossibility. A probability of 1 indicates certainty.

Example 6 *The probability that a STAT 7100 student is both absent for class and present in class is 0.*

Example 7 *The probability that today is Monday and today is Tuesday is 0.*

Example 8 *The probability that today is Monday or today is Tuesday is $\frac{2}{7}$.*

Example 9 *When rolling a die, the probability that the number will be even or odd is 1.*

Example 10 *The probability that someone will post something insensitive or offensive to the internet today is 1.*

Problem 4 *An experiment consist of flipping a fair coin twice. Compute the probabilities of the following events.*

A - Exactly one head is observed.
B - At least one head is observed.
C - No tails are observed.
Construct a tree diagram and sample space for this experiment.

HH	TH
HT	TT

$P(A) = \frac{2}{4} = \frac{1}{2} = 0.5$, $P(B) = 3/4 = 0.75$ and $P(C) = 1/4 = 0.25$.

Problem 5 *Repeat for $n = 3$ flips of the coin*

Problem 6 *Repeat for $n = 10$ flips of the coin but let's not draw the tree diagram or list all elements of the sample space.*

Example 11 *A pair of fair dice is rolled. Compute the probabilities of the following events.*

A - The sum of the two dice is 7.
B - The sum of the two dice is 5
C - The sum of the two dice is an even number.

$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$	$(6,1)$
$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$	$(6,2)$
$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$	$(5,3)$	$(6,3)$
$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	$(5,4)$	$(6,4)$
$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$	$(5,5)$	$(6,5)$
$(1,6)$	$(2,6)$	$(3,6)$	$(4,6)$	$(5,6)$	$(6,6)$

$P(A) = \frac{6}{36} = 0.166\,67$, $P(B) = \frac{4}{36} = 0.111\,11$ and $P(C) = \frac{18}{36} = 0.5$.

Exercise 1 *For any pair of dice, must there always exist the same number of odd sums as even sums?*

Exercise 2 *A single six-sided die is rolled. If the number face up is even, then record the number and the experiment is over. If the number face up is odd, then record the number and the die is rolled again. The second number is also recorded and the experiment is over. How many different outcomes are possible? Are they all equally likely?*

Exercise 3 *Consider the class for each of the 60 students in MATH 1107/01 as given in the table below.*

Class	*Frequency*
Freshman	*15*
Sophomore	*21*
Junior	*5*
Senior	*19*

What is the probability that a randomly selected student is a senior?

What is the probability that a randomly selected student is a freshman or sophomore?

These last two examples illustrate two rules of probability.

Theorem 3 *For any event A, $P(\bar{A}) = 1 - P(A)$.*

Theorem 4 *If A and B are disjoint (mutually exclusive) events then $P(A$ or $B) = P(A) + P(B)$.*

Example 12 *Pick a single card from a deck. What is the probability that you select an Ace or an 8?*
These are disjoint events. There are four of each rank in a deck of cards. Thus, the probability that you select an Ace or an 8 is $\frac{4}{52} + \frac{4}{52} = \frac{8}{52} = 0.153\,85$.

Example 13 *Pick a single card from a deck. What is the probability that you do not select an Ace?* $1 - P(\bar{A}) = 1 - \frac{1}{4} = \frac{3}{4}$.

Exercise 4 *Pick a single card from a deck. What is the probability that you select a club or a diamond?*

Exercise 5 *A study at a local bar found people of various ages playing games. Each patron participates in exactly one game.*

	21-29	30-39	40-49	50 and older	Total
Darts	4	12	15	6	37
Pool	8	17	16	11	52
Karaoke	17	5	0	1	23
Total	29	34	31	18	112

Find the probability that a randomly selected person...

1. is playing pool;

2. is 30-39;

3. is playing pool or singing karaoke;

4. is 21-29 or 40-49.

Of course there are times when we want to determine $P(A \text{ or } B)$ when A and B are not disjoint. In such a case we must employ the general additions rule.

3 The General Addition Rule

For any events A and B, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

Example 14 *A card is selected at random. What is the probability it is a club or a 3?* $P(\textit{club or 3}) = P(\textit{club}) + P(3) - P(3 \textit{ of clubs}) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{4}{13} = 0.307\,69$.

Exercise 6 *A card is selected at random. What is the probability it is a face card or a heart?*

Exercise 7 *Roll a pair of dice. Compute the probability that*

1. the sum is 7 or 8;

2. the sum of the dice is at least 10?

3. the sum of the dice is at least 3?

4. the sum is 5 or a 2 appears on at least one die.

Exercise 8 *A study at a local bar found people of various ages playing games.*

	21-29	30-39	40-49	50 and older	Total
Darts	4	12	15	6	37
Pool	8	17	16	11	52
Karaoke	17	5	0	1	23
Total	29	34	31	18	112

Find the probability that a randomly selected person...

1. is playing pool or darts;

2. is 30-39 or throwing darts;

3. is playing pool or 50 and older;

4. is 21-29 or singing Karaoke.

4 Unusual Outcomes

We've discussed data points that are unusual in a sample or population (outliers). When is an event considered unusual? The rule-of-thumb is that an event whose probability is less than or equal to $5\% = .05$ is considered unusual. When randomly selection a student from MATH 1107/01, would it be unusual to pick a junior? Since $P(\text{pick a junior}) = \frac{5}{60} = \frac{1}{12} = .083$ then no it would not be unusual.

5 Exercises

1. Navidi Section 2.1: 1, 2 (you may assume the numbers are different colors so that you can distinguish between all the 1s and the pair of 2s), 3-10, 12-15, 19