Section 2.3: The Multiplication Rule and Conditional Probability

Since the size of a sample space grows so quickly we want to continue our search for rules of that allow us to compute the probabilities of complex events. When thinking about what happens with combinations of outcomes, things are simplified if the individual trials are independent.

Definition 1 *Two events are independent if the outcome of one event doesn't influence or change the likelihood of the outcome of the other event.*

Example 1 *Coin flips are independent. The coin does not remember its sequence of flips; the chance of heads or tails is always constant at* $p = \frac{1}{2}$.

Example 2 *From ajc.com on 1/6/2017,*
"As the metro area faces another dire winter forecast, it's hard to forget that only three years ago we let 2.6 inches of snow knock us on our collective backside, turning Atlanta into national laughingstock. We called it "SnowJam '14." Not to be confused with "Snow Jam '82" where nearly the same thing happened, or "Snowpocalypse '11" which had been so recent that some leaders figured the region was statistically safe from another snow debacle for at least a decade."

Example 3 *The Phillies chance of winning the World Series in 2017 and the health of Phillies pitchers are dependent events. If any pitcher suffers a serious injury and is out for the season, the Phillies chance of winning the World Series goes down.*

Example 4 *The Phillies chance of winning the World Series in 2017 and the health of pitchers of their opponents are dependent events. If any pitcher suffers a serious injury and is out for the season, their team's chance of winning goes down and the Phillies chance of winning the World Series goes up.*

Example 5 *The Phillies chance of winning the World Series in 2017 and the health of Dr. DeMaio's pitching arm are independent events. No matter what happens to Dr. DeMaio, the Phillies chances of winning the World Series are completely unaffected.*

Theorem 1 *Multiplication Rule: For two independent events A and B, the probability that both A and B occur is the product of the probabilities of the two events.*

$$P(A \text{ and } B) = P(A) \times P(B)$$

Example 6 *Approximately 85% of all human beings are right-handed. What is the probability that three randomly selected people are all right-handed?* $p = .85 \times .85 \times .85 = .85^3 = 0.614\,13$.

Example 7 *Ignoring ambidextrous people, What is the probability that two randomly selected people are all left-handed?* $p = .15^2 = 0.0225$.

Exercise 1 *Shaquille O'Neal's lifetime free throw percentage is .527. Shaq is fouled on a three-point shot.*
What is the probability that he makes all three free throws?

What is the probability that he makes none of the free throws?

Exercise 2 *A box contains 3 white balls, 4 red balls and 5 black balls. A ball is picked, its color recorded and returned to the box. Another ball is then selected and its color recorded.*

Remark 2 *Since we put the 1st ball back into the box before selecting another, we are making* ***selections with replacement****. Doing so makes subsequent selections independent events.*
Find the probability that 2 black balls are selected.

Find the probability that 2 balls of the same color are selected.

Example 8 *A box contains 3 white balls, 4 red balls and 5 black balls. Four balls are picked with replacement.*
Find the probability no red balls are selected.

Find the probability that the fourth ball selected is the first occurrence of the color white?

In most situations where we want to find a probability, we'll use the rules in combination. A good thing to remember is that it can be easier to work with the complement of the event we're really interested in. This is almost always the case when you encounter the *phrase* **at least one**. The event of **at least one** is equivalent to **one or more**. Note that if event A is the event of **at least one** then the complement of A is **none**.

Example 9 *A die is independently rolled 8 times. What is the probability that the number 2 appears at least once?* $p = 1 - \left(\frac{5}{6}\right)^8 = 0.767\,43$

Example 10 *Shaquille O'Neal's lifetime free throw percentage is .527. Shaq is fouled on a three-point shot. What is the probability that he makes at east one of his three free throws?*

Exercise 3 *According to Nielson Media Research, 30% of all televisions are tuned to NFL Monday Night Football when it is televised. Assuming that this show is being broadcast and that the televisions are randomly selected, find the probability that at least 1 of 15 televisions is tuned to NFL Monday Night Football.*

A requirement of the multiplication rule is that events are independent. Naturally, this will not always be the case. In order to compute the probability of A and B when they are not independent events we rely on conditional probabilities. When we want the probability of an event from a conditional distribution, we write $P(B|A)$ and say "the probability of B given A." A probability that takes into account a given condition is called a **conditional probability**.

Theorem 3 *General Multiplication Rule: For any two events A and B, the probability that both A and B occur is the .*

$$P(A \text{ and } B) = P(A) \times P(B|A)$$

Exercise 4 *A study at a local bar found people of various ages playing games.*

	21-29	**30-39**	**40-49**	**50 and older**	**Total**
Darts	4	12	15	6	37
Pool	8	17	16	11	52
Karaoke	17	5	0	1	23
Total	29	34	31	18	112

Find the probability that a randomly selected person...

1. Plays darts.
2. Is 21-29.
3. Is 21-29 given that they are playing darts.
4. Is 21-29 given that they are singing karaoke.
5. Is singing karaoke given that they are 21-29.
6. Is 30-39 and playing pool.
7. Is playing pool given that they are 30-39.

A pair of dice is thrown one at a time. Let A be the event that the sum of 9 is rolled. Let B be the event that the first die thrown is a 2. Let C be the event that the first die thrown is a 5. Let D be the event that the sum of 7 is rolled.

1. What is the probability the sum of the dice is 9?
2. What is the probability the sum of the dice is 9, given that the first die rolled is 2?
3. What is the probability the sum of the dice is 9, given that the first die rolled is 5?
4. Are events A and B independent? Are events A and C independent?
5. What is the probability the sum of the dice is 7?
6. What is the probability the sum of the dice is 7, given that the first die rolled is 2?
7. What is the probability the sum of the dice is 7, given that the first die rolled is 5?
8. Are events D and B independent? Are events D and C independent?

Exercise 5 *A box contains 3 white balls, 4 red balls and 5 black balls. A ball is picked, its color recorded and set aside. Another ball is then selected and its color recorded.*

Remark 4 *In this case, we did not return the 1st ball back to the box before selecting another. We are now making* ***selections without replacement****. Doing so makes subsequent selections dependent or conditional events.*
Find the probability that 2 black balls are selected.

Find the probability that 2 balls of the same color are selected.

Find the probability that the second ball selected is the first occurrence of the color white?

Example 11 *In a room with 23 people, what is the probability that at least one pair of them share a birthday?*

The key phrase "at least one" appears and motivates us to use complements. Thus,

$$P(\text{at least one pair share a b-day}) = 1 - P\left(\text{none share a b-day}\right).$$

The first person can have any birthday. The second person can have any birthday other than the first. The third person can have any birthday other than the first two and so.

$$\begin{aligned} & P(\text{at least one pair share a b-day}) \\ = \; & 1 - P\left(\text{none share a b-day}\right) \\ = \; & 1 - \frac{366}{366} * \frac{365}{366} * \frac{364}{366} * ... * \frac{366-22}{366} \\ = \; & 1 - \prod_{i=0}^{22} \frac{366-i}{366} \\ = \; & 0.506\,32. \end{aligned}$$

Knowing that

$$P(A \text{ and } B) = P(A) \times P(B|A)$$

allows us to rearrange terms and see that

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}.$$

Example 12 *For residents of Atlanta, 35% are fans of the Braves, 45% are fans of the Falcons and 13% are fans of both the Braves and the Falcons.*

1. Given that a person is a Falcons fan, what is the probability that they are a Braves fan? $P(B|F) = \frac{P(B \text{ and } F)}{P(F)} = \frac{.13}{.45} = 0.288\,89$.

2. What is the probability that a person is neither a Braves fan nor a Falcons fan? $1 - P(B \text{ or } F) = 1 - (.35 + .45 - .13) = 0.33$.

3. Given that a person is not a Braves fan, what is the probability that they are not a Falcons fan? $P(\bar{F}|\bar{B}) = \frac{P(\bar{B} \text{ and } \bar{F})}{P(\bar{B})} = \frac{.33}{.65} = 0.507\,69$.

1 Bayes' Theorem

In its simplest form, Bayes' Theorem permits you to compute one conditional probability of events A and B if you already know the other one.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A) * P(A)}{P(B)}.$$

Example 13 *Kate and Clint each shoot at a target. Kate will hit the target 60% of the time. Clint's attention span tends to be inconsistent. He needs focus to want to hit the target. When Kate misses the target, Clint hits the target 45% of the time. When Kate hits the target, Clint's focus is vastly improved and he will then hit the target 90% of the time. Given that Clint hits the target, what is the probability that Kate also hit the target? According to Bayes' Theorem,* $P(K|C) = \dfrac{P(K \cap C)}{P(C)}$. *What is* $P(C)$? *It changes based on what Kate does. We can think of* $P(C)$ *as depending on Kate missing or hitting the target. Thus,* $P(C) = .4 * .45 + .6 * .9 = 0.72$. *Thus,* $P(K|C) = \dfrac{P(K \cap C)}{P(C)} = \dfrac{P(C|K) * P(K)}{P(C)} = \dfrac{.9 * .6}{.72} = 0.75$

Extending to a more complex form, Bayes' Theorem is a natural extension of both the multiplication rule and disjoint addition rule. Consider events $A_1, A_2, ..., A_k$ which are pairwise disjoint. If $\bigcup_{i=1}^{k} A_i = S$ then we call the collection of events **exhaustive**. We've really just partitioned the sample space S into disjoint parts and make repeated applications of the addition rule for disjoint sets.

Theorem 5 *(Bayes') Let $A_1, A_2, ..., A_k$ be pairwise disjoint and exhaustive over S. For any outcome B with non-zero probability in S*

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{k} P(A_j)P(B|A_j)}.$$

Example 14 *Let's assume 0.025 of the population is infected by Covid 19. Let A_1 represent a patient with Covid-19 and Let A_2 represent a patient without Covid-19. Let B represent a positive test result. If we know that a person with Covid-19 will test positive 60% of the time and a person without Covid-19 will test positive 5% of the time, what is the probability that a person with a positive test has Covid-19?*
We want to find $P(A_1|B)$. Since we do not know $P(B)$ we cannot compute directly $P(A_i|B) = \frac{P(A_i \cap B)}{P(B)}$. According to Bayes' Theorem

$$\begin{aligned} P(A_1|B) &= \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)} \\ &= \frac{0.025 * 0.6}{0.025 * 0.6 + 0.975 * 0.05} \\ &= 0.235\,29 \end{aligned}$$

Example 15 *Having reacquired the Infinity Gauntlet, Thanos is once again about to snap his fingers and eliminate some of the Earth's population of human beings. Let L represent the event of surviving. While there will be an element of chance to the eliminations, Thanos has stacked the deck in favor of his long term survival. Thanos himself will live. Super-heroes (S) have a 30% survival rate, super-villains (V) have a 45% survival rate and all others (O) have a 55% survival rate. It is estimated that 3% of the population are super-heroes, 12% are super-villains leaving a grand total of 85% as other. Meanwhile you are in the maternity ward staring down at your new child who has yet to be classified as hero, villain or other. Given that your child survives, what is the likelihood they will be a super-villain?*

$$\begin{aligned} & P(V|L) \\ =& \frac{P(V)P(L|V)}{P(V)P(L|V) + P(S)P(L|S) + P(O)P(L|O)} \\ =& \frac{.12 * .45}{.12 * .45 + .03 * .3 + .85 * .55} \\ =& 0.10179 \end{aligned}$$

2 Exercises

1. Navidi: 1-3, 5, 7-10, 14, 17-19, 24, 27, 32, 33