Section 4.3: The Poisson Distribution

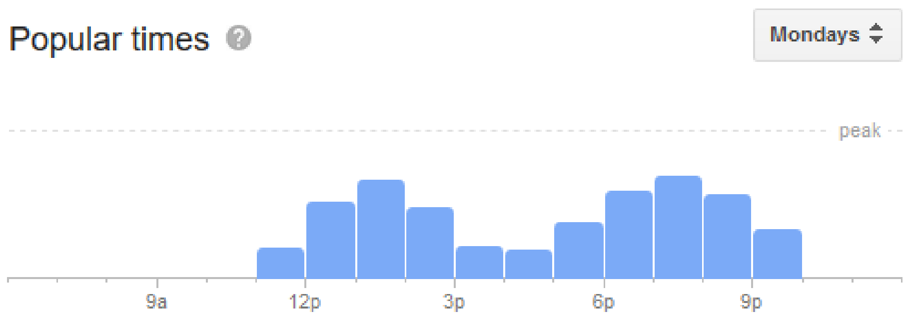
The Poisson distribution counts the number of occurrences of a particular event in a fixed unit of measurement.

Definition 1 *An experiment has a* ***Poisson Probability Distribution*** *if and only if*

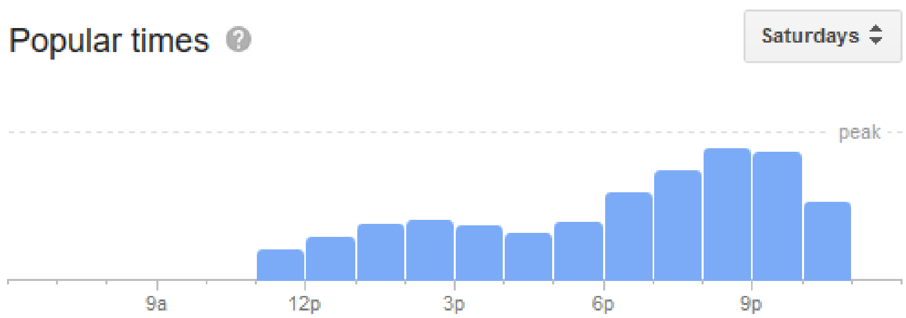
1. *the probability that a single event occurs in any fixed unit of measurement is the same for all intervals;*
2. *the mean number of events λ that occur in any interval is independent of the number that occur in any other interval.*

Remark 2 *The poisson random variable is denoted by $X\tilde{}Poisson(\lambda)$.*

Example 3 *Customer dining at Rusan's on Barrett Pkwy on Mondays does not follow a Poisson distribution since the average number of customers is not fixed throughout all time periods of the day.*



Example 4 *Customer dining at Rusan's on Barrett Pkwy on from 8 PM to 9 PM during the week does not follow a Poisson distribution since the average number of customers is not fixed throughout all days.*



Example 5 *Customer dining at Rusan's on Barrett Pkwy on from 1 PM to 2 PM on Monday does follow a Poisson distribution since the average number of customers is fixed throughout all Mondays.*

Example 6 *Suppose the average number of underfilled 12 ounce cans of cola in a 12 pack is* $\lambda = .01$ *and cans are independently filled. This process follows a Poisson distribution.*

Theorem 7 *If an experiment has a Poisson probability distribution where* λ *is the average number of occurrences in any fixed unit of measurement then*

1. *the probability of* x *occurrences in a fixed period of measurement is* $p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$*;*

2. *the mean or expected number of occurrences in a fixed period of measurement is* $\mu = \lambda$*;*

3. *and* $V(X) = \lambda$ *and thus, the standard deviation is* $\sigma = \sqrt{\lambda}$.

Remark 8 *The Poisson Distribution is a discrete random variable (since each event happens or it does not) but does not arise from counting principles as our previous examples have. It arises from the expansion of* $e^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$*. If we multiple both sides by* $e^{-\lambda}$ *we see that* $1 = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$ *and by pushing the constant* $e^{-\lambda}$ *through the summation we get a probability distribution since* $1 = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!}$ *and since* $\frac{e^{-\lambda}\lambda^x}{x!} > 0$*, it must be the case that* $\frac{e^{-\lambda}\lambda^x}{x!} < 1$.

Remark 9 *On the TI-83/84 series of calculators,* ***poissonpdf***(λ, k) *computes the probability of* x *occurrences in any fixed unit of measurement . The command* ***poissontcdf***(λ, x) *computes the probability of* 0 *or* 1 *or* 2 *or...or* x *occurrences in any fixed unit of measurement*

Example 10 *Suppose the average number of customers from 1 PM to 2 PM at Rusan's on Monday is* $\lambda = 25$.

1. *What is the standard deviation for the average number of customers from 1 PM to 2 PM at Rusan's on Monday?* $\sigma = \sqrt{25} = 5$.

2. *What is the probability that exactly 20 customers dine from 1 PM to 2 PM at Rusan's next Monday?*
   $p(20) = \frac{e^{-25}25^{20}}{20!} = 5.1917 \times 10^{-2}$

3. *What is the probability that no more than 4 customers dine from 1 PM to 2 PM at Rusan's next Monday?*
   $p(0) + p(1) + p(2) + p(3) + p(4) = \frac{e^{-25}25^0}{0!} + \frac{e^{-25}25^1}{1!} + \frac{e^{-25}25^2}{2!} + \frac{e^{-25}25^3}{3!} + \frac{e^{-25}25^4}{4!} = 2.6691 \times 10^{-7}$

4. *What is the probability that between 23 and 27 customers dine from 1 PM to 2 PM at Rusan's next Monday?*
   $p(23) + p(24) + p(25) + p(26) + p(27) = \frac{e^{-25}25^{23}}{23!} + \frac{e^{-25}25^{24}}{24!} + \frac{e^{-25}25^{25}}{25!} + \frac{e^{-25}25^{26}}{26!} + \frac{e^{-25}25^{27}}{27!} = 0.382\,65.$

Problem 11 *On the TV show, Criminal Minds, Dr. Spencer Reid has an average of 5 socially awkward moments per episode.*

1. *Find the mean and standard deviation for the number of socially awkward moments per episode.*

2. *Find the probability that Dr. Reid has no socially awkward moments in tonight's episode.*

What if we don't know λ? Can we estimate it? Of course we can.

Problem 12 *Paul bakes pecan cookies in batches of 50. You eat the first five cookies out of the tent while counting pecans. These five cookies have 1,3,2,4 and 0 pecans. Estimate* λ *for the number of pecans per cookie. We use* $\hat{\lambda} = \frac{X}{t}$ *where* X *is the total number of events observed in period* t*. In this case that is* $\frac{1+3+2+4+0}{5} = 2$*.*

Problem 13 *How much uncertainty do we expect in counting pecans in Paul's cookies? In general,* $\sigma_{\hat{\lambda}} = \sqrt{\frac{\hat{\lambda}}{t}}$ *and here specifically* $\sigma_{\hat{\lambda}} = \sqrt{\frac{2}{5}} = 0.632\,46$*.*

0.1 Exercises

Navidi 4.3: 1-9, 14, 17a-d