Section 4.4: Some Other Discrete Distributions

1 The Hypergeometric Distribution

The geometric probability distribution looks for the first success where selections are made with replacement (or the sample size is less than 5% of the population size). The hypergeometric distribution addresses the experiments where selections are made without replacement.

Definition 1 *An experiment consisting of repeated trials has a **Hypergeometric Probability Distribution** if and only if*

1. *the population is finite of size N;*
2. *there are only two outcomes, success or failure, and the population contains M successes;*
3. *each sample of size n is equally likely.*

Knowing an experiment has a hypergeometric distribution provides the following formulae.

Theorem 2 *If an experiment has a hypergeometric probability distribution with parameters n, N and M then*

1. *the probability of exactly x successes is $p(x) = \dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$;*
2. *the mean or expected value of the number of successes*
$$E(X) = n * \frac{M}{N}$$
which by letting $p = \frac{M}{N}$ transforms into
$$E(X) = np$$
3. *and the variance of the number of successes is*
$$V(X) = \left(\frac{N-n}{N-1}\right) n \frac{M}{N} \left(1 - \frac{M}{N}\right)$$
which by letting $p = \frac{M}{N}$ transforms into
$$V(X) = \left(\frac{N-n}{N-1}\right) np(1-p).$$

Proof. Note that since selections are made without replacement, we are choosing unordered subsets.

1. There are clearly $\binom{N}{n}$ different ways to select a sample of size n from a population of size N. With M successes, we can select x of those in $\binom{M}{x}$ ways. If there are M successes in the population then there must be $N-M$ failures in the population. We need to pick $n-x$ of those for the sample. This can be done in $\binom{N-M}{n-x}$ ways. Thus, the probability of exactly x successes is $p(x) = \dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$.

2. According to the STAT 7010 textbook, the moment generating function for the hypergeometric probability distribution is "more trouble than it worth here."

3. Ditto from 2.

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Example 3 *Fred's horror movie collection consists of 10 films that have a sense of humor and 8 that do not. Fred plans a Saturday night triple feature and randomly selects three horror films from his collection. What is the probability that Fred picks two films that have a sense of humor and 1 that does not? Here* $N = 18$*,* $n = 3, M = 10$ *and* $x = 2$*. So,*

$$\begin{aligned} p(x) &= \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \\ & \quad \textit{So, } p(2) \\ &= \frac{\binom{10}{2}\binom{8}{1}}{\binom{18}{3}} \\ &= 0.441\,18. \end{aligned}$$

Remark 4 *I think it is much easier to understand the hypergeometric formula from the perspective of success and failure than using so many variables. Your mileage may vary. Furthermore we could consider picking a film with no sense of humor as a success and compute* $p(1)$*. This yields* $p(1) = \dfrac{\binom{8}{1}\binom{10}{2}}{\binom{18}{3}} = 0.441\,18$*. Naturally this is the same value since multiplication is a commutative operation.*

Example 5 *When Fred randomly selects three horror films from his collection what is* $E(X), V(X)$ *and* σ *for the number of films that have a sense of humor? Note that* $p = \frac{M}{N} = \frac{10}{18}$ *for selecting a film with a sense of humor. So,*

$$\begin{aligned} E(X) &= np \\ &= 3 * \frac{10}{18} \\ &= 1.666\,7 \end{aligned}$$

and

$$\begin{aligned} V(X) &= \left(\frac{N-n}{N-1}\right) np(1-p) \\ &= \left(\frac{18-3}{18-1}\right) * 3 * \frac{10}{18} * (1 - \frac{10}{18}) \\ &= 0.653\,59 \end{aligned}$$

Thus, $\sigma = \sqrt{0.653\,59} = 0.808\,45$.

Example 6 *Back to Jason's CD collection that consists of five different rock CDs, three different jazz CDs, two different blues CDs, two different classical CDs and a single folk CD. Jason is planning a trip and randomly selects four CDs. What is the probability that Jason selects three rock CDs? We now see this as a hypergeometric problem where selecting a rock CD is considered a success. So,*

$$\begin{aligned} p(x) &= \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \\ & \textit{So, } p(3) \\ &= \frac{\binom{5}{3}\binom{8}{1}}{\binom{13}{4}} \\ &= 0.111\,89. \end{aligned}$$

What are $E(X), V(X)$ *and* σ *for the number of rock CDs that Jason selects?* $E(X) = np = 4 * \frac{5}{13} = 1.538\,5$.

$$\begin{aligned} V(X) &= \left(\frac{N-n}{N-1}\right) np(1-p) \\ &= \left(\frac{13-4}{13-1}\right) 4 * \frac{5}{13}(1 - \frac{5}{13}) \\ &= 0.710\,06 \end{aligned}$$

Hence, $\sigma = \sqrt{0.710\,06} = 0.842\,65$.

Example 7 *Jason also packs his 6 can cooler to capacity for the road trip at random from a fridge with 27 beers and 13 sodas.*

1. What is the probability that he randomly picks three beers and three sodas? $p(3) = \frac{\binom{27}{3}\binom{13}{3}}{\binom{40}{6}} = 0.217\,94$.

2. What is the probability that the number of beers is larger than the number of sodas?

$$\begin{aligned} & p(4) + p(5) + p(6) \\ &= \frac{\binom{27}{4}\binom{13}{2}}{\binom{40}{6}} + \frac{\binom{27}{5}\binom{13}{1}}{\binom{40}{6}} + \frac{\binom{27}{6}\binom{13}{0}}{\binom{40}{6}} \\ &= 0.707\,17. \end{aligned}$$

Remark 8 *Note that the binomial probability distribution and the hypergeometric distribution have the same expected value np. The* $V(X)$ *of the two distributions differ only by* $\left(\frac{N-n}{N-1}\right)$. *This is referred to as the finite population correction factor. Note that when n is small compared to* N, $\left(\frac{N-n}{N-1}\right)$ *is approximately 1. Mathematically we see that when n is small compared to N it is very reasonable to use the binomial distribution to approximate the hypergeometric distribution (assume selections are made with replacement.*

2 The Geometric Distribution

The necessary properties of a geometric probability distribution are similar to those of the binomial probability distribution.

Definition 9 *An experiment consisting of repeated trials has a* ***Geometric Probability Distribution*** *if and only if*

1. *there are only two outcomes for each trial (success or failure) and the probability* p *of success is fixed;*
2. *the trials are independent.*

The target of the geometric distribution is the number of trials until the first success. Knowing an experiment has a geometric distribution provides the following formulas.

Theorem 10 *If an experiment has a geometric probability distribution with fixed probability of success* p *then*

1. *the probability that the first success occurs on trial* k *is* $p(k) = p\,(1-p)^{k-1}$;
2. *the mean or expected value number of trials until the first success is* $\mu = \frac{1}{p}$;
3. *and the standard deviation is* $\sigma = \frac{\sqrt{(1-p)}}{p}$.

Remark 11 *On the TI-83/84 series of calculators,* ***geometpdf***(p,k) *computes the probability of requiring* k *trials for the initial success. The command* ***geometcdf***(p,k) *computes the probability of requiring* 1 *or* 2 *or...or* k *trials for the initial success.*

Example 12 *Recall that Shaquille O'Neal's lifetime free throw percentage is* 0.527. *Assume that free throw attempts are independent of one another.*

1. Find the mean and standard deviation for the number of attempts needed before making a free throw.
   $\mu = \frac{1}{p} = \frac{1}{0.527} = 1.8975$ and $\sigma = \frac{\sqrt{(1-0.527)}}{0.527} = 1.305$

2. What is the probability Shaq makes his first free throw on his fourth attempt?
   $p(4) = 0.527\,(1-0.527)^{4-1} = 5.\,576\,9 \times 10^{-2}$

3. What is the probability it takes Shaq no more than three attempts before he makes his first free throw?
   $p(x \leq 3) = p(1)+p(2)+p(3) = 0.527\,(1-0.527)^{1-1}+0.527\,(1-0.527)^{2-1}+0.527\,(1-0.527)^{3-1} = 0.894\,18$

4. What is the probability Shaq makes his first free throw on his fourth, fifth or sixth attempt?
   $p(4)+p(5)+p(6) = 0.527\,(1-0.527)^{4-1}+0.527\,(1-0.527)^{5-1}+0.527\,(1-0.527)^{6-1} = 9.\,462\,5 \times 10^{-2}$

5. What is the probability it takes Shaq at least three attempts before he makes his first free throw?
   $p(x \geq 3) = 1-p(1)+p(2) = 1-(0.527\,(1-0.527)^{1-1}+0.527\,(1-0.527)^{2-1}) = 0.223\,73$

Problem 13 *John intends to roll a pair of dice until he gets a sum of 12.*

1. *How many times do you expect John to roll the dice?*

2. *What is the probability John rolls the dice no more than 10 times?*

3. *What is the probability John requires between 30 and 40 attempts before hitting the sum of 12?*

Problem 14 *In 2017, 71% of all full-time KSU undergraduates received some type of need-based financial aid (https://www.usnews.com/best-colleges/kennesaw-state-university-1577). What is the average number of undergraduates that must be randomly selected to find someone without need-based financial aid?*

3 The Negative Binomial Distribution

Definition 15 *An experiment consisting of repeated trials has a* ***Negative Binomial Probability Distribution*** *if and only if*

1. *there are only two outcomes for each trial (success or failure) and the probability p of success is fixed;*

2. *the trials are independent;*

3. *the experiment continues until there are r successes.*

The random variable X is the number of failures that precede the r^{th} success. Knowing an experiment has a negative binomial distribution provides the following formulae.

1. The probability of x failures before the r^{th} success is $p(x) = \binom{x+r-1}{r-1} p^r (1-p)^x$;Note that $\binom{x+r-1}{x} = \binom{x+r-1}{r-1}$;

2. $E(X) = \frac{r(1-p)}{p}$;

3. $V(X) = \frac{r(1-p)}{p^2}$.

1. If we have x failures before the r^{th} success then we have $x + r$ trials. Furthermore, the $x + r^{th}$ is a success. The remaining $r - 1$ successes can occur anywhere is the first $x + r - 1$ trials. Thus, we need to pick which of those trials yields the $r - 1$ successes. We can do so in $\binom{x+r-1}{r-1}$ ways. Then we must attain those r successes and fail the remaining x times. Hence, $p(x) = \binom{x+r-1}{r-1} p^r (1 - p)^x$.

2. We let $E(X) = \frac{r(1-p)}{p}$ and $V(X) = \frac{r(1-p)}{p^2}$ stand without proof.

Example 16 *A pair of fair dice is rolled.*

1. What is the likelihood we roll the dice 13 times before seeing a sum of 7 twice? Using $p(x) = \binom{x+r-1}{r-1} p^r (1-p)^x$, $p(11) = \binom{11+2-1}{2-1} * \frac{1}{6}^2 * (1 - \frac{1}{6})^{11} = 4.4863 \times 10^{-2} = 0.0448$

2. What is the expected number of failures before rolling two sums of 7? $E(X) = \frac{r(1-p)}{p} = \frac{2(1-\frac{1}{6})}{\frac{1}{6}} = 10.0$

3. What is the variance and standard deviation of the number of failures before rolling two sums of 7? $V(X) = \frac{2(1-\frac{1}{6})}{\frac{1}{6}^2} = 60.0$ and $\sigma = \sqrt{60} = 7.7460$

4. What is the likelihood we roll the dice no more than 5 times before seeing

a sum of 7 twice?

$$\begin{aligned}
& \sum_{x=0}^{3} p(x) \\
= & \sum_{x=0}^{3} \binom{x+r-1}{r-1} p^r (1-p)^x \\
= & \sum_{x=0}^{3} \binom{x+2-1}{2-1} \frac{1}{6}^2 (1-\frac{1}{6})^x \\
= & \sum_{x=0}^{3} \binom{x+1}{1} \frac{1}{36} (\frac{5}{6})^x \\
= & \sum_{x=0}^{3} (x+1) \frac{1}{36} (\frac{5}{6})^x \\
= & (1\frac{1}{36}(\frac{5}{6})^0 + (2)\frac{1}{36}(\frac{5}{6})^1 + (3)\frac{1}{36}(\frac{5}{6})^2 + (4)\frac{1}{36}(\frac{5}{6})^3 \\
= & 0.196\,24
\end{aligned}$$

5. What is the likelihood we roll the dice at least 6 times before seeing a sum of 7 twice? $1 - \sum_{x=0}^{3} p(x) = 1 - 0.196\,24 = 0.803\,76$.

4 Exercises

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