1 Computations

1. $10! = 3628\,800$
$\frac{500!}{502!} = \frac{1}{251\,502} = 3.\,976\,1 \times 10^{-6}$
$\binom{18}{4} = 3060$
$\frac{\binom{2n}{2}}{\binom{2n}{3}} = \frac{\frac{2n(2n-1)}{2}}{\frac{2n(2n-1)(2n-2)}{3*2}} = \frac{2n(2n-1)}{2} * \frac{3*2}{2n(2n-1)(2n-2)} = \frac{3}{2n-2}$

2. Without calculating a final answer, show $16! = 14!5!2!$.

$16! = 16 * 15 * 14! = 4 * 2 * 2 * 5 * 3 * 14! = 14! * 5 * 4 * 3 * 2 * 2 = 14!5!2!$.
3. $\sum_{i=1}^{7} 2i - 4 = 52$
$\sum_{i=24}^{79} i^2 = 163\,156$
Note that $\sum_{i=1}^{n} i^2 = \frac{1}{6}n\,(2n+1)\,(n+1)$ and thus $\sum_{i=24}^{79} i^2 = \sum_{i=1}^{79} i^2 - \sum_{i=1}^{23} i^2 = 167\,480 - 4324 = 163\,156$.

$\prod_{i=-533}^{278} \left(i^3 - 1\right) = 0$
Since 1 is an integer between -533 and 278 the product contains the term $1^3 - 1 = 0$ and is 0.
4. $\sum_{i=1}^{n} 2^i \binom{n}{i} = \sum_{i=1}^{n} 2^i * 1^{n-i} \binom{n}{i} = (2+1)^n = 3^n$ by the binomial theorem.

5. List the members of the set $S = \left\{x | x \in Z^+, 50 \leq x^3 \leq 150\right\}$.
$S = \{4, 5\}$

6. Construct $P(A)$ for $A = \{*, a, 3\}$.
$P(A) = \{\emptyset, \{*\}, \{a\}, \{3\}, \{*, a\}, \{*, 3\}, \{a, 3\}, \{*, a, 3\}\}$
Compute $|P(A)|$ for $A = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31\}$.
$|P(A)| = 2^{11} = 2048$

7. Give an example of sets A and B such that B is a proper subset of A and $|A| = |B|$.
Let $A = Z$ and let $B = Z^+$.

8. Give an example of sets A, B and C such that $|A| = |B| = |C| = \aleph_0$ where $|A - B| = \aleph_0$ but $|A - C| = \emptyset$.

Let $A = \mathbb{Z}^+, B = 2\mathbb{Z}^+$ and $C = \mathbb{Z}$.

9. Use the binomial theorem to expand $(3x-2)^4$ into polynomial form. You must show all details.

$(3x-2)^4 = \binom{4}{4}(3x)^4(-2)^{4-4} + \binom{4}{3}(3x)^3(-2)^{4-3} + \binom{4}{2}(3x)^2(-2)^{4-2} + \binom{4}{1}(3x)^1(-2)^{4-1} + \binom{4}{0}(3x)^0(-2)^{4-0} = 81x^4 - 216x^3 + 216x^2 - 96x + 16$

10. Find the coefficient of x^{10} in the expansion of

(a) $(2x-4)^{15}$; $\binom{15}{10}(2x)^{10}(-4)^5 = -3148\,873\,728x^{10}$

(b) $\left(4x^3 - 3\right)^{12}$; The coefficient is 0 since no integer power of $4x^3$ will yield x^{10}.

(c) $\left(8x^5 - 3\right)^5$. $\binom{5}{2}\left(8x^5\right)^2(-3)^3 = -17\,280x^{10}$

11. Let $g_0 = 1$. Let $g_n = 2^{g_{n-1}}$ for $n \geq 1$. Compute g_1, g_2, g_3 and g_4.

$g_1 = 2^{g_0} = 2^1 = 2$

$g_2 = 2^{g_1} = 2^2 = 4$

$g_3 = 2^{g_2} = 2^4 = 16$

$g_4 = 2^{g_3} = 2^{16} = 65\,536$

Give a recursive definition of the set of positive integer powers of 5.

Let $P_1 = 5$. Let $P_n = 5P_{n-1}$ for $n \geq 2$.

12. How many positive integers not exceeding 1000 are divisible by 6, 9 or 15? $\left\lfloor \frac{1000}{6} \right\rfloor + \left\lfloor \frac{1000}{9} \right\rfloor + \left\lfloor \frac{1000}{15} \right\rfloor - \left\lfloor \frac{1000}{18} \right\rfloor - \left\lfloor \frac{1000}{30} \right\rfloor - \left\lfloor \frac{1000}{45} \right\rfloor + \left\lfloor \frac{1000}{90} \right\rfloor = 244$

13. Let $U = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{1,4,5,8,9\}$ and $B = \{2,4,5,6,9,10\}$. List the members of the following sets.

i. $A \cup B = \{1,2,4,5,6,8,9,10\}$

ii. $A \cap B = \{4,5,9\}$

iii. $\overline{\bar{A} \cup B}$

$\bar{A} \cup B = \{2,3,4,5,6,7,9,10\}$ so $\overline{\bar{A} \cup B} = \{1,8\}$

iv. $A - B = \{1,8\}$

v. $A \oplus B = \{1,2,6,8,10\}$

2 Problems

1. (10 points) Shot in the back while playing poker, Wild Bill Hickok's final hand was a pair of aces and a pair of eights, now known as the dead man's hand. How many different dead man's hands exist? $\frac{\binom{4}{2}^2 * 44}{\binom{52}{5}} = \frac{33}{54\,145} = 6.094\,7 \times 10^{-4}$

2. (10 points) Using a standard deck of 52 cards (with no jokers) determine the probability of a full house.

$\frac{13\binom{4}{3}12\binom{4}{2}}{\binom{52}{5}} = \frac{6}{4165} = 1.4406 \times 10^{-3}$

3. (15 points) Using a standard deck of 52 cards (with no jokers) where each of the four deuces is considered a wild card, which hand should win: a five of a kind or a royal flush?

i) The five of a kind can be made with 1, 2, 3 or 4 deuces. Also note that you cannot achieve a five of a kind with deuces. Using these four disjoint cases, there exist

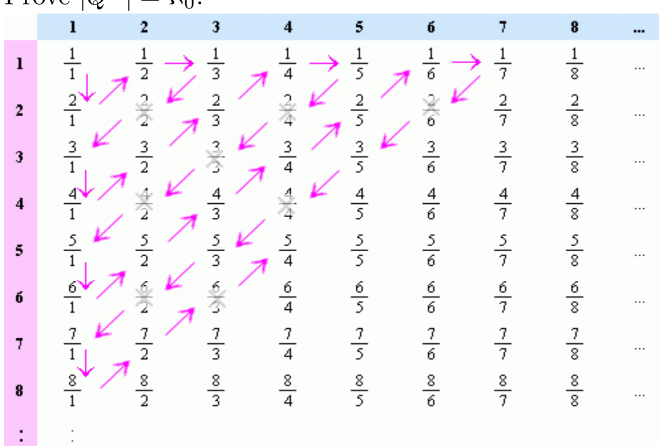
$12 * \binom{8}{5} = 672$ different five of a kinds.

ii) The royal flush can be made with 0, 1, 2, 3 or 4 deuces.

$4 * \binom{9}{5} = 504$ different royal flush hands.

Since the royal flush is the scarcer hand, it should be the more powerful hand.

4. Prove $|Q^+| = \aleph_0$.



5. Use mathematical induction to prove $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in Z^+$.

I. Show that $S(1)$ is true. L.H.S. $\sum_{i=1}^{1} i^3 = 1^3 = 1$. R.H.S. $\frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$. Thus, $S(1)$ is true.

II. Assume $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ and show $\sum_{i=1}^{n+1} i^3 = \frac{(n+1)^2(n+2)^2}{4}$.

$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3$ which by the inductive hypothesis is $\frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} = \frac{(n+1)^2\left[n^2+4(n+1)\right]}{4} = \frac{(n+1)^2\left[n^2+4n+4\right]}{4} = \frac{(n+1)^2(n+2)^2}{4}$.

6. (10 points) Use induction to prove 3 divides $n^3+2n-15$ for all $n \in Z^+$.
   I. $\frac{1^3+2*1-15}{3} = \frac{1+2-15}{3} = \frac{-12}{3} = -4 \in Z$.
   II. Assume $\frac{n^3+2n-15}{3} \in Z$. Show $\frac{(n+1)^3+2(n+1)-15}{3} \in Z$
   $\frac{(n+1)^3+2(n+1)-15}{3} = \frac{\left(n^3+3n^2+3n+1\right)+(2n+2)-15}{3} = \frac{n^3+2n-15}{3} + \frac{3n^2+3n+3}{3}$ which by the inductive assumption is $int + \frac{3n^2+3n+3}{3} = int + \frac{3\left(n^2+n+1\right)}{3} = int + n^2 + n + 1 = int$.

7. Use induction to prove $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$ for the Fibonacci sequence and $n \in Z^+$.

   First show that the base case of $n = 1$ hold true.

   R.H.S. $\sum_{i=1}^{1} f_i^2 = f_1^2 = 1^2 = 1$

   L.H.S. $f_1 f_{1+1} = f_1 f_2 = 1 * 1 = 1$

   Assume the statement is true for n : $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$. Show that the statement is true for $n+1$: $\sum_{i=1}^{n+1} f_i^2 = f_{n+1} f_{n+2}$.

   Note that $\sum_{i=1}^{n+1} f_i^2 = \sum_{i=1}^{n} f_i^2 + f_{n+1}^2$ which by the inductive hypothesis is $f_n f_{n+1} + f_{n+1}^2 = f_{n+1}(f_n + f_{n+1}) = f_{n+1} f_{n+2}$.

8. For sets A, B and C, let $A \oplus C = B \oplus C$. Use contradiction to prove $A = B$.
   Assume $A \neq B$. Without loss of generality we can say that if $A \neq B$, then there exists $x \in A$ such that $x \notin B$. We now have two cases.
   1. If $x \in C$, then $x \notin A \oplus C$ but $x \in B \oplus C$. This is a contradiction since $A \oplus C = B \oplus C$.
   2. If $x \notin C$, then $x \in A \oplus C$ but $x \notin B \oplus C$. This is a contradiction since $A \oplus C = B \oplus C$.
   In either case a contradiction arises. Thus, the assumption is wrong and $A = B$.

9. Use a combinatorial proof to show $\binom{4n}{2} = 4\binom{n}{2} + 6n^2$ for $n \in Z^+$.
   Let $A = \{1, 2, ..., n, ..., 4n\}$ and let S be the collection of all subsets of A of size 2. On the one hand, it is clear that $|S| = \binom{4n}{2}$. One the other hand partition A into four disjoint subsets. Let $B = \{1, 2, ...n\}$,

$C = \{n+1, n+2, ...2n\}$, $D = \{2n+1, 2n+2, ...3n\}$ and $E = \{3n+1, ..., 4n\}$. We can select a set of two elements of A by selecting two elements from one of the sets B, C, D, or E. This can be done in $4\binom{n}{2}$ ways. Or we can select two of the sets B, C, D, or E and select one element from each. This can be done in $\binom{4}{2}n^2 = 6n^2$ ways. Thus, $|S| = 4\binom{n}{2} + 6n^2$. We've counted the same set S in two different ways and $\binom{4n}{2} = 4\binom{n}{2} + 6n^2$ for $n \in Z^+$.

10. A woman has 11 close friends. Find the number of ways she can invite 5 of them to dinner where:
i. There are no restrictions. $\binom{11}{5} = 462$
ii. Two of the friends are newlyweds and will not attend separately.
There are two disjoint cases: the couple is invited or not. $\binom{9}{3} + \binom{9}{5} = 210$
iii. Two of the friends are divorced from each other and will not attend together. Here we have two cases: neither are invited or exactly one of the two is invited. $\binom{9}{5} + 2\binom{9}{4} = 378$

11. Two married couples, two single men and one single woman sit in a row of seven consecutive seats. How many ways can they be seated
i. with no restrictions? $7! = 5040$
ii. alternating genders? With 4 men and 3 women the only way to alternate gender is with a MWMWMWM arrangement. Hence there exist $4 * 3 * 3 * 2 * 2 * 1 * 1 = 4! * 3! = 144$ different arrangements.
iii. such that the women are all consecutive? There are 3! ways to arrange the women in a consecutive block of seats. Now we have 4 men and 1 block of women to arrange. There are a total of $3! * 5! = 720$ legal arrangements.
iv. such that spouses sit next to one another. As in part iii. we need to arrange each married couple in a block. Each married couple can be arranged in $2! = 2$ ways. Now we are left with 5 types of objects to arrange in seats. There are a total of $2! * 2! * 5! = 480$ different arrangements.

12. A social club consists of 10 pairs of twins, some identical, some fraternal. Of these 20 members, 13 are women and 7 are men.
i. How many ways can this club select a social coordinator and treasurer if these roles must be served by different club members? $20 * 19 = 380$
ii. How many ways can this club select a social coordinator and treasurer if these roles must be served by club members of opposite gender? $2 * 13 * 7 = 182$
iii. How many ways can this club form a six person committee to coordinate the club trip to the annual national meeting of twins? Note that each person on this committee is of equal rank and power. $\binom{20}{6} = 38\,760$
iv. How many ways can this club form a six person committee that consists of three twin pairs to coordinate the club trip to the annual national meeting of twins? Note that each person on this committee is of equal rank and power. $\binom{10}{3} = 120$
v. How many ways can this club form a six person committee that consists of six club members but no twin pairs to coordinate the club trip

to the annual national meeting of twins? Note that each person on this committee is of equal rank and power. $\binom{10}{6} * 2^6 = 13\,440$

13. At a particular university, a student's password consists of five lowercase letters such that no letters are repeated, the first letter is a letter from the student's first name and the last letter is any vowel (no y's allowed), not necessarily from the student's first name. How many different passwords can Mark create?
Pick the last letter first. The problem here is that we don't know if the last letter is an 'a' or another vowel. So, there are two disjoint cases: the last letter is an 'a' or the last letter is not an 'a.' Pick the last letter first, then the first from Marks' name and then the rest in order. By the sum rule this can be done in $1 * 3 * 24 * 23 * 22 + 4 * 4 * 24 * 23 * 22 = 230\,736$

14. A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels and plain bagels. How many ways are there to choose
i. 6 bagels; $n = 8, k = 6$ unordered set with repetition: $\binom{8+6-1}{6} = 1716$
ii. 2 dozen bagels; $n = 8, k = 24$ unordered set with repetition: $\binom{8+24-1}{24} = 2629\,575$
iii. a dozen bagels with at least one of each kind? If we pick one of each flavor to start we really only make a selection of 4 additional bagels. $n = 8, k = 4$ unordered set with repetition: $\binom{8+4-1}{4} = 330$

15. Susan buys an economy pack of fifty pens. The pens are identical except for color. There are ten of each of five different colors.
i. How many different ways can Susan select four pens of different colors to take to work? $\binom{5}{4} = 5$
ii. How many different ways can Susan select ten pens to take to work? $\binom{5+10-1}{10} = 1001$
iii. How many different ways can Susan select twelve pens to take to work with at least one of each color pen? $\binom{7+5-1}{7} = 330$
iv. How many different ways can Susan select six pens to take to work with at least two different colors in the mix? $\binom{5+6-1}{6} - 5 = 205$
v. How many different ways can Susan select twelve pens to take to work? $\binom{5+12-1}{12} - 5 - 5 * 4 = 1795$

16. True or False: $\binom{n}{j+k} = \binom{n}{j} + \binom{n}{k}$ for all $n, j, k \in \mathbb{Z}^+$. If true, prove it. If false, provide a counterexample. False! Let $n = 10$ and $j = k = 1$. Now $\binom{n}{j+k} = \binom{10}{2} = 45$ while $\binom{n}{j} + \binom{n}{k} = \binom{10}{1} + \binom{10}{1} = 20$.

17. True or False: $j! + k! = (j + k)!$ for all $j, k \in \mathbb{Z}^+$. If true, prove it. If false, provide a counterexample. False. Let $j = k = 2$. Now $j! + k! = 2! + 2! = 4$ while $(j + k)! = 4! = 24$.