Math 3322 Final Exam
DeMaio Spring 2009

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. Where appropriate, simplify all answers to a single decimal expansion.

1. (30 points) Determine each of the following.
(a) $|\mathbb{Z}^+ - \mathbb{R}| = 0$
(b) $\frac{305!}{303!} = 305 * 304 = 92\,720$
(c) $\binom{37}{15} = 9364\,199\,760$
(d) $\sum_{i=1}^{123} i = \frac{123*124}{2} = 7626$
(e) For $x > 1$, $\left\lfloor 3.7 + \left\lceil \frac{x}{x+1} \right\rceil + \left\lceil \frac{x+1}{x} \right\rceil \right\rfloor = \lfloor 3.7 + 1 + 2 \rfloor = 6$
(f) The coefficient of x^7 in the polynomial expansion of $(6 - 3x)^{10}$. The coefficient if $\binom{10}{7} 6^3 (-3x)^7 = -56\,687\,040x^7$

2. (20 points) Susan buys an economy pack of sixty pens. The pens are identical except for color. There are ten of each of six different colors.
(a) How many different ways can Susan select four pens of different colors to take to work? $\binom{6}{4} = 15$
(b) How many different ways can Susan select ten pens to take to work? $\binom{6+10-1}{10} = 3003$
(c) How many different ways can Susan select ten pens to take to work with at least one of each color? $\binom{6+4-1}{4} = 126$
(d) How many different ways can Susan select twelve pens to take to work? **Hint!** This answer not just the solution of (b) with different numbers! $\binom{6+12-1}{12} - 6 - 6 * 5 = 6152$

3. (25 points) Let $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$.
(a) How many non-empty subsets does S have? $2^8 - 1 = 255$
(b) How many subsets of S have no odd numbers as members? $2^4 = 16$
(c) How many subsets of S have exactly 4 elements? $\binom{8}{4} = 70$
(d) How many four digit numbers can be made using the digits of S if a digit may be used repeatedly? Before you answer, ask yourself if 0 can be a leading digit. $7 * 8^3 = 3584$
(e) How many even four digit numbers can be made using the digits of S if a digit may be used only once? **Be careful!** This reminds me of the problem regarding Mark's password.
The difficulty here is if 0 is the last digit or not. We'll break this down into the two cases where 0 is the last digit and 0 is not the last digit. We'll select the last digit first and then the rest in their natural order. We have $1 * 7 * 6 * 5 + 3 * 6 * 6 * 5 = 750$ different even four digit numbers from S with no repeated digits.

4. (5 points) How many positive integers not exceeding 1000 are divisible by 6, 9 or 15? $\left\lfloor \frac{1000}{6} \right\rfloor + \left\lfloor \frac{1000}{9} \right\rfloor + \left\lfloor \frac{1000}{15} \right\rfloor - \left\lfloor \frac{1000}{18} \right\rfloor - \left\lfloor \frac{1000}{30} \right\rfloor - \left\lfloor \frac{1000}{45} \right\rfloor + \left\lfloor \frac{1000}{90} \right\rfloor = 244$

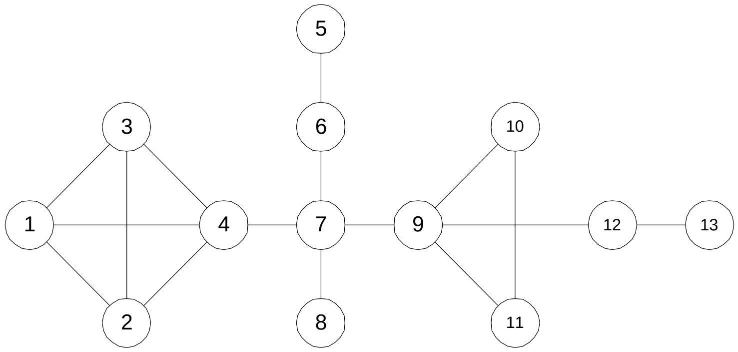
5. (5 points) A small local business has 85 offices. Of these offices, 70 have a computer, 25 have a fax machine and 33 have a paper shredder. There are 20 offices that have both a computer and a fax machine, 27 offices that have both a computer and paper shredder and 15 offices with both a fax machine and paper shredder. There are 12 offices that have a computer, fax machine and paper shredder. How many offices have none of computer, fax machine or shredder? Note that $70+25+33-20-27-15+12 = 78$ have at least one of the items. Thus, $85 - 78 = 7$ have none.

6. (10 points) (a)Use a combinatorial proof to show $\binom{4n}{2} = 4\binom{n}{2} + 6n^2$.
Let $A = \{1, 2, ..., 4n\}$ and let S be the collection of all subsets of size 2 from A. On the one hand, $|S| = \binom{4n}{2}$. On the other hand, let $B = \{1, 2, ..., n\}$, $C = \{n+1, n+2, ..., 2n\}$, $D = \{2n+1, 2n+2, ..., 3n\}$ and $E = \{3n + 1, 3n + 2, ..., 4n\}$. How can we select two elements from A relative to B, C, D and E? We can select two elements from a single set in $4\binom{n}{2}$ ways or we can select one each set from any pair of sets in $\binom{4}{2}n^2 = 6n^2$ ways. Thus, $|S| = 4\binom{n}{2} + 6n^2$. Put the two together and $\binom{4n}{2} = 4\binom{n}{2} + 6n^2$.

(5 points) (b) Without proof, write an identity for $\binom{kn}{2}$ similar to (a).
$\binom{kn}{2} = k\binom{n}{2} + \binom{k}{2}n^2$

7. (10 points) Use induction to show $\frac{2^{3n}-22}{7}$ is an integer for $n \in \mathbb{Z}^+$.
   1. Show $S(1)$ is true. Note $\frac{2^{3*1}-22}{7} = -2 \in \mathbb{Z}$.
   2. Show if $S(n)$ is true and then $S(n+1)$ is also true. Assume $\frac{2^{3n}-22}{7} \in \mathbb{Z}$ and show $\frac{2^{3(n+1)}-22}{7} \in \mathbb{Z}$.
   $\frac{2^{3(n+1)}-22}{7} = \frac{2^{3n+3}-22}{7} = \frac{8*2^{3n}-22}{7} = \frac{7*2^{3n}}{7} + \frac{2^{3n}-22}{7} = 2^{3n} + \frac{2^{3n}-22}{7} = int + int = int.$

8. (18 points) Consider the following graph $G = (V, E)$.



   i. $n = 13$
   ii. $e = 16$
   iii. $N(7) = \{4, 6, 8, 9\}$
   iv. $\deg(9) = 4$
   v. The set of all pendants of G is {5,8,13}
   vi. The set of all isolated vertices of G is $\emptyset$.

9. (10 points) Let $G = (V, E)$ be a graph with at least 2 vertices. Use the pigeonhole principle to show there exist vertices $u, v \in V$ such that $\deg(u) = \deg(v)$.
   See class notes.

10. (5 points) Suppose a graph has 150 edges, 30 vertices of degree 4, and all others of degree 3. How many vertices does the graph have?
   $\sum_{v \in V} \deg(v) = 2e.$
   On one hand, $\sum_{v \in V} \deg(v) = 2 * 150 = 300$.On the other hand, let x be the number of vertices of degree 3 and $\sum_{v \in V} \deg(v) = 4 * 30 + 3x$. Hence, $300 = 120 + 3x$: Solution is: $x = 60$. The graph has $30 + 60 = 90$ vertices.