A Graph Theoretic Summation of the Cubes of the First $n$ Integers

Joe DeMaio  
Kennesaw State University  
Kennesaw, GA 30144  
jDMAio@kennesaw.edu

Andy Lightcap  
Kennesaw State University  
Kennesaw, GA 30144  
andy.lightcap@gmail.com

The complete graph $K_{n+1}$ contains $n+1$ vertices and $\binom{n+1}{2}$ edges. Iteratively building the complete graph $K_{n+1}$ by introducing vertices one at a time and counting the new edges incident to the new vertex provides a combinatorial proof that $\sum_{i=1}^{n} i = \binom{n+1}{2}$.

![Figure 1: $\sum_{i=1}^{n} i = \binom{n+1}{2}$](image)

Since $\sum_{i=1}^{n} i^3 = \left(\binom{n+1}{2}\right)^2$ it seems natural to look for a combinatorial proof that also uses graphs. Consider the complete bipartite graph $K_{\binom{n+1}{2}, \binom{n+1}{2}}$ that contains $2\binom{n+1}{2}$ vertices and $\binom{n+1}{2}^2$ edges. As before, we will count the new edges incident to newly introduced vertices in $n$ stages. At stage $i$ we introduce $i$ new vertices to each side of the graph and count the edges incident to these new vertices. Since $\sum_{i=1}^{n} i = \binom{n+1}{2}$ this process enumerates all the edges in $K_{\binom{n+1}{2}, \binom{n+1}{2}}$. New vertices on one side are adjacent only to vertices on the other side. When just considering the edges between the new vertices, the subgraph $K_{i,i}$ immediately appears with $i^2$ edges. It turns out that these $i^2$ edges along with the additional edges constructed between a new vertex on one side and an old vertex on the other side will always total $i^3$ new edges. This shows that $\sum_{i=1}^{n} i^3 = \left(\binom{n+1}{2}\right)^2$.

In order to see that we always introduce $i^3$ new edges at stage $i$, we will partition the new edges into complete bipartite graphs. At stage $i$, there exist
\[ i = \frac{i(i-1)}{2} \]

previously introduced vertices on each side of the graph and the
new vertices on each side are labeled \( (\frac{i}{2}) + 1, (\frac{i}{2}) + 2, \ldots, (\frac{i}{2}) + i = (\frac{i+1}{2}) \). The
partition of these edges into complete bipartite graphs depends upon the parity
of \( i \). Figure 2 illustrates these stages for \( n = 5 \). To prevent a deluge of edges in
the graph, a complete bipartite graph such as \( K_{2,4} \) is represented as 

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
\end{array}
\]

When \( i \) is odd, the new edges quickly form \( i \) disjoint copies of \( K_{i,i} \). For odd
\( i \) we partition the old vertices into \( \frac{i-1}{2} \) sets of \( i \) vertices for each side. Both
sets of \( i \) new vertices are adjacent to each of the \( \frac{i-1}{2} \) sets of \( i \) vertices on the
other side. This yields \( 2(\frac{i-1}{2}) = i - 1 \) additional copies of \( K_{i,i} \). Along with
the initial copy of \( K_{i,i} \) on only the new vertices, we have \( i \) copies of \( K_{i,i} \) for a
total of \( i^3 \) new edges.

When \( i \) is even, we have to work a bit harder. For even \( i \), we partition the
old vertices on each side into \( \frac{i}{2} - 1 \) sets of \( i \) vertices and one set of \( \frac{i}{2} \) vertices.
This yields \( 2 \left( \frac{i}{2} - 1 \right) \) copies of \( K_{i,i} \) and two copies of \( K_{\frac{i}{2},i} \) for \( 2 \left( \frac{i}{2} - 1 \right) i^2 + 2 \frac{i}{2} i = i^3 - i^2 \) edges. As before, with the original \( K_{i,i} \) between the sets of new vertices,
the total once again is \( i^3 \) new edges.

References

of the first \( n \) integers, \( The College Mathematics Journal 38 \) (2007) 296.