Math 3322 Midterm
DeMaio Fall 2010

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (35 points) Compute the following.
   i. $\sum_{i=1}^{164} i^2 = 1483\,790$
   ii. $\sum_{i=1}^{45} i^3 = 1071\,225$
   iii. $\left\lfloor \frac{1}{2} \left\lceil \frac{3}{2} \right\rceil \right\rfloor = 1$
   iv. $\frac{(98+3)!}{98!} = 999\,900$
   v. $|P(A)|$ where $A = \{1, \{1\}, \{2,3\}, \{2,3,4\}\}$. $|P(A)| = 2^4 = 16$.
   vi. the first 10 terms of the sequence whose n^{th} term is the largest integer k such that $k! \leq n$.
   $1, 2, 2, 2, 2, 3, 3, 3, 3, 3$
   vii. A theater concession counter offers four different sizes of drinks and eight different choices of beverages. How many different ways can a drink be ordered?
   $4 * 8 = 32$

2. (5 points) Find the domain and range of the function that assigns to each bit string twice the number of zeroes in that string.
   The domain is the collection of all bit strings. The range is the set of even non-negative integers.

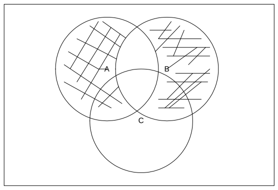
3. (10 points) Let $g(n) = 4g(n-1) + 2$ for $n \geq 2$ where $g(1) = 5$. Compute $g(2)$, $g(3)$, and $g(4)$.
   $g(2) = 4 * 5 + 2 = 22$
   $g(3) = 4 * 22 + 2 = 90$
   $g(4) = 4 * 90 + 2 = 362$

4. (5 points) Write a recursive definition of the odd positive integers.
   Let $O_n = O_{n-1} + 2$ where $O_1 = 1$.

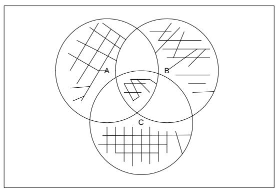
5. (10 points) Use induction to prove $\dfrac{3^{4n+8} - 11^{n+3}}{10} \in \mathbb{Z}$ for all $n \in \mathbb{Z}^+$.
   Show that $S(1)$ is true. Note that $\dfrac{3^{4*1+8} - 11^{1+3}}{10} = 51\,680 \in \mathbb{Z}$. Now show that if $S(n)$ is true then $S(n+1)$ is also true. Assume $\dfrac{3^{4n+8} - 11^{n+3}}{10} \in \mathbb{Z}$ and show $\dfrac{3^{4(n+1)+8} - 11^{n+1+3}}{10} = \dfrac{3^{4n+12} - 11^{n+4}}{10} \in \mathbb{Z}$. We see that $\dfrac{3^{4n+12} - 11^{n+4}}{10} = \dfrac{3^4 * 3^{4n+8} - 11 * 11^{n+3}}{10} = \dfrac{81 * 3^{4n+8} - 11 * 11^{n+3}}{10} = \dfrac{3^{4n+8} - 11^{n+3}}{10} + \dfrac{80 * 3^{4n+8}}{10} - \dfrac{10 * 11^{n+3}}{10} = int$ by inductive assumption $+ 8 * 3^{4n+8} - 11^{n+3} = int + int + int = int$.

6. (10 points) A collection of seven distinct coins will be arranged from left to right. There are four heads face up and three tails face up.
   i. How many different ways can the coins be arranged from left to right? $7! = 5040$
   ii. How many different ways can the coins be arranged from left to right if there can be no consecutive heads? $4 * 3 * 3 * 2 * 2 * 1 * 1 = 144$
   iii. How many different ways can the coins be arranged from left to right if all heads must be consecutive and all tails must be consecutive? $2! * 4! * 3! = 288$
   iv. How many different ways can the coins be arranged from left to right if all heads must be consecutive? $4! * 4! = 576$

7. (15 points) Let A and B be sets.
i. Shade $(A \oplus B) \oplus C$ in a Venn diagram.



$A \oplus B$



$(A \oplus B) \oplus C$

ii. Prove $A \subseteq (A \oplus B) \oplus B$.
Let $x \in A$. Either $x \in B$ or $x \notin B$. This leads to two cases to consider.
1. If $x \in B$ then $x \notin (A \oplus B)$ since x would be in both A and B. However, since $x \in B$ and $x \notin (A \oplus B)$ then $x \in (A \oplus B) \oplus B$.
2. If $x \notin B$ then $x \in (A \oplus B)$ since $x \in A$ but not in B. Furthermore since $x \in (A \oplus B)$ but not in B then $x \in (A \oplus B) \oplus B$.
In either case, $x \in (A \oplus B) \oplus B$ and $A \subseteq (A \oplus B) \oplus B$.

8. (20 points) State the definition of the Fibonacci sequence.
$F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ where $F_0 = 0$ and $F_1 = 1$.
Compute the following:
$F_2 = 1$ $\quad F_3 = 2$ $\quad F_4 = 3$ $\quad F_5 = 5$ $\quad F_6 = 8$
Prove $\sum\limits_{i=0}^{n} F_{3i} = \frac{F_{3n+2}-1}{2}$ for all positive integers n.
First show that $S(1)$ is true.
L.H.S. $\sum\limits_{i=0}^{1} F_{3i} = F_0 + F_3 = 0 + 2 = 2$
R.H.S. $\frac{F_{3*1+2}-1}{2} = \frac{F_5-1}{2} = \frac{5-1}{2} = 2$
Thus, $S(1)$ is true. Now assume $\sum\limits_{i=0}^{n} F_{3i} = \frac{F_{3n+2}-1}{2}$ and show $\sum\limits_{i=0}^{n+1} F_{3i} = \frac{F_{3(n+1)+2}-1}{2} = \frac{F_{3n+5}-1}{2}$.
$\sum\limits_{i=0}^{n+1} F_{3i} = \sum\limits_{i=0}^{n} F_{3i} + F_{3(n+1)} = \sum\limits_{i=0}^{n} F_{3i} + F_{3n+3} = \frac{F_{3n+2}-1}{2} + F_{3n+3} = \frac{F_{3n+2}-1+2F_{3n+3}}{2} = \frac{F_{3n+2}+F_{3n+3}+F_{3n+3}-1}{2} = \frac{F_{3n+4}+F_{3n+3}-1}{2} = \frac{F_{3n+5}-1}{2}$.

9. Let A_n be the **set** of all bit strings of length n that contain no consecutive 1's.
i. (1 point) $A_1 = \{0, 1\}$
ii (2 points). $A_2 = \{00, 01, 10\}$
iii (3 points). $A_3 = \{000, 100, 010, 001, 101\}$
iv. (4 points) Conjecture, without proof, a formula for $|A_n|$.
It appears that $|A_n| = F_{n+2}$