Math 3322 Midterm
DeMaio Spring 2011

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (35 points) Compute the following.
   i. $\sum_{i=1}^{74} i^2 = 137\,825$
   ii. $\sum_{i=10}^{45} i^3 = 1069\,200$
   iii. For $\frac{1}{2} < x < 1$, $\left\lfloor \left\lceil \frac{x}{x^2} \right\rceil \right\rfloor = 2$
   iv. $\frac{(98+3)!}{98!} = 999\,900$
   v. $|P(A)|$ where $A = \{1, \{1\}, \{2,3\}, \{2,3,\{4\}\}, \mathbb{R}, \emptyset\}$
   $|P(A)| = 2^6 = 64$
   vi. the first 10 terms of the sequence whose n^{th} term is the smallest integer k such that $k! > n$.
   $2, 3, 3, 3, 3, 4, 4, 4, 4, 4$
   vii. For any set A, $A \oplus A = \emptyset$

2. (5 points) Find the domain and range of the function that assigns to each student in Math 3322 (a class that meets twice a week for 15 weeks) twice the number of classes missed.
   The domain is the collection of all students in Math 3322. The range is the set of integers from 0 to 30.

3. (10 points) Let $g(n) = 4g(n-1) + g(n-2)$ for $n \geq 2$ where $g(0) = 1$ and $g(1) = 1$. Compute $g(2)$, $g(3)$, and $g(4)$.
   $g(2) = 4 * g(1) + g(0) = 4 * 1 + 1 = 5$
   $g(3) = 4 * g(2) + g(1) = 4 * 5 + 1 = 21$
   $g(4) = 4 * g(3) + g(2) = 4 * 21 + 5 = 89$

4. (5 points) Write a recursive definition of the odd positive integers.
   Let $O_n = O_{n-1} + 2$ for $n \geq 2$ where $O_1 = 1$.

5. (10 points) Use induction to prove $\dfrac{3^{4n+8} - 11^{n+3}}{10} \in \mathbb{Z}$ for all $n \in \mathbb{Z}^+$.
   Show that $S(1)$ is true. Note that $\dfrac{3^{4*1+8} - 11^{1+3}}{10} = 51\,680 \in \mathbb{Z}$. Now show that if $S(n)$ is true then $S(n+1)$ is also true. Assume $\dfrac{3^{4n+8} - 11^{n+3}}{10} \in \mathbb{Z}$ and show $\dfrac{3^{4(n+1)+8} - 11^{n+1+3}}{10} = \dfrac{3^{4n+12} - 11^{n+4}}{10} \in \mathbb{Z}$. We see that $\dfrac{3^{4n+12} - 11^{n+4}}{10} = \dfrac{3^4 * 3^{4n+8} - 11 * 11^{n+3}}{10} = \dfrac{81 * 3^{4n+8} - 11 * 11^{n+3}}{10} = \dfrac{3^{4n+8} - 11^{n+3}}{10} + \dfrac{80 * 3^{4n+8}}{10} - \dfrac{10 * 11^{n+3}}{10} = int$ by inductive assumption $+ 8 * 3^{4n+8} - 11^{n+3} = int + int + int = int$.

6. (10 points) True or False: $A \oplus B = (A - B) \cap (B - A)$? If true, prove. If false, provide a counterexample.
   False. Let $A = \{1,2\}$ and $B = \{2,3\}$. $A \oplus B = \{1,3\}$ while $(A-B) \cap (B-A) = \{1\} \cap \{3\} = \emptyset$. In fact, for every A and B, $(A-B) \cap (B-A) = \emptyset$ so all you have to do is select $A \neq B$.

7. (10 points) Prove that any amount of postage of at least 24 cents can be formed by using only 5 cent and 7 cent stamps.
   We need five base cases.
   $24 = 2 * 7 + 2 * 5$
   $25 = 5 * 5$
   $26 = 3 * 7 + 5$
   $27 = 7 + 4 * 5$
   $28 = 4 * 7$

Now assume that every amount of postage from $24, 25, 26, 27, 28, ..., n$ cents for $n \geq 28$ can be formed by using only 5 cent and 7 cent stamps. Show that $n+1$ cents worth of postage can be formed by using only 5 cent and 7 cent stamps. Since $n+1 = 5+n-4$ we can add a 5 cent stamp to our solution for $n-4$ cents worth of postage. How do we know a solution exists for $n-4$ cents worth of postage? Since $n \geq 28$, $n-4 \geq 24$ and we know a solution exists by our inductive assumption.

8. (15 points) Prove $\sum_{i=0}^{n} F_{3i} = \frac{F_{3n+2}-1}{2}$ for all positive integers n.

   First show that $S(1)$ is true.

   L.H.S. $\sum_{i=0}^{1} F_{3i} = F_0 + F_3 = 0 + 2 = 2$

   R.H.S. $\frac{F_{3*1+2}-1}{2} = \frac{F_5-1}{2} = \frac{5-1}{2} = 2$

   Thus, $S(1)$ is true. Now assume $\sum_{i=0}^{n} F_{3i} = \frac{F_{3n+2}-1}{2}$ and show $\sum_{i=0}^{n+1} F_{3i} = \frac{F_{3(n+1)+2}-1}{2} = \frac{F_{3n+5}-1}{2}$.

   $\sum_{i=0}^{n+1} F_{3i} = \sum_{i=0}^{n} F_{3i} + F_{3(n+1)} = \sum_{i=0}^{n} F_{3i} + F_{3n+3} = \frac{F_{3n+2}-1}{2} + F_{3n+3} = \frac{F_{3n+2}-1+2F_{3n+3}}{2} = \frac{F_{3n+2}+F_{3n+3}+F_{3n+3}-1}{2} = \frac{F_{3n+4}+F_{3n+3}-1}{2} = \frac{F_{3n+5}-1}{2}$.

9. (15 points) Someone claims that $r_1, r_2, r_3, ...$ is an ordered listing of all reals in $(0, 1)$. Produce $r \in (0, 1)$ such that $r \neq r_i$ for all $i \in \mathbb{Z}^+$. Once again, see class notes.