Math 3322 Quiz I
DeMaio Spring 2010

Name________________________________________________

**Instructions.** Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (10 points) Let $S = \{x | x \in \mathbb{Z}^+, x \text{ is a solution of } \left(x^2 - 1\right)(2x - 7)(3x - 12) = 0\}$. List the elements of $S$.
   The solutions of $\left(x^2 - 1\right)(2x - 7)(3x - 12) = 0$ are $-1, 1, 3.5$ and $4$. Of those, only 1 and 4 are elements of $\mathbb{Z}^+$. Thus, $S = \{1, 4\}$

2. (10 points) Use set builder notation to define the set of all rational numbers.
   $\mathbb{Q} = \{\frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0\}$

3. (10 points) Let $A = \{1, 2, 3\}$ and $B = \{\alpha, \beta\}$. Construct $A \times B$.
   $A \times B = \{(1, \alpha), (1, \beta), (2, \alpha), (2, \beta), (3, \alpha), (3, \beta)\}$

4. (15 points) Compute the cardinality of each of the following sets.
   i. $\emptyset$; $|\emptyset| = 0$;
   ii. $B = \{1, \beta, c, \{1\}, \{\{\alpha, \beta, \gamma, \delta\}\}, \mathbb{Z}, \mathbb{Z}^+\}$;
   $|B| = 7$
   iii. $P(C)$ for $C = \{1, 2, 3, ..., 15\}$.
   $|P(C)| = 2^{|C|} = 2^{15} = 32\,768$

5. (30 points) Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{1, 5\}$, $B = \{2, 3, 7\}$ and $C = \{1, 7\}$. List the members of each of the following sets.
   i. $A \cap B = \emptyset$
   ii. $A \cap C = \{1\}$
   iii. $\overline{A \cup B} =$
   $A \cup B = \{1, 2, 3, 5, 7\}$ so $\overline{A \cup B} = \{4, 6\}$
   iv. $\overline{A} \cup \overline{B} = \{1, 2, 3, 4, 5, 6, 7\} = U$
   v. $(A \cup B) \cap C = \{1, 7\} = C$
   vi. $A \oplus C = \{5, 7\}$

6. Let $A$ and $B$ be sets.
   i. (5 points) State the definition of $A$ is a **subset** of $B$.
   $A$ is a subset of $B$ if and only if for every $x \in A$, $x \in B$.
   ii. (5 points) What is the difference in meaning of $A \subseteq B$ versus $A \subset B$?
   The notation $A \subseteq B$ allows $A = B$ while $A \subset B$ forces $A$ to be a proper subset of $B$.

   iii. (10 points) Let $A$ and $B$ be sets. Prove $(B - A) \cup (C - A) \subseteq (B \cup C) - A$.
   Let $x \in (B - A) \cup (C - A)$. This forces $x \in (B - A)$ or $x \in (C - A)$. Note that in either case, $x \notin A$ since that would remove $x$ from the set difference! Furthermore, $x \in (B - A)$ or $x \in (C - A)$ forces $x$ to be a member of $B$ or $C$. That fact demonstrates that $x \in B \cup C$. So, we have that $x \in B \cup C$ and $x \notin A$. So, $x \in (B \cup C) - A$.

7. (15 points) Prove $\sqrt{13}$ is irrational. Assume $\sqrt{13}$ is rational. This allows us to write $\sqrt{13} = \frac{a}{b}$ where $a, b \in \mathbb{Z}, b \neq 0$. Furthermore we can assume that $a$ and $b$ share no factors since we can always just reduce the fraction to its simplest terms. Now, if $\sqrt{13} = \frac{a}{b}$ then $13 = \frac{a^2}{b^2}$ and $a^2 = 13b^2$. So 13 divides $a^2$. Since 13 is prime we know that 13 must also divide $a$ and we can write $a = 13k$ for some $k \in \mathbb{Z}$. Now we rewrite $a^2 = 13b^2$ as $(13k)^2 = 13b^2$ and see that $b^2 = \frac{(13k)^2}{13} = 13k^2$. So 13 divides $b^2$. Once again, since 13 is prime we know that 13 must also divide $b$. This, however, is a contradiction since $a$ and $b$ share no common factors. So, the assumption that $\sqrt{13}$ is rational is false and we've shown that $\sqrt{13}$ is irrational.