Math 3322 Quiz I Key
DeMaio Summer 2009

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (10 points) List the members of the sets
   i. $S = \{x | x \leq 100 \text{ and } \sqrt{x} \in \mathbb{Z}\}$.
   $S = \{0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$
   ii. $S = \{x | x \in \mathbb{Z} \text{ such that } x^2 = 2\}$.
   $S = \emptyset$

2. (10 points) Use set builder notation to give a description of $S = \{..., -15, -10, -5, 0, 5, 10, 15, ...\}$.
   $S = \{5x | x \in \mathbb{Z}\}$

3. (10 points) Find $P(A)$ for $A = \{1, \mathbb{R}, \emptyset\}$.
   $P(A) = \{\emptyset, \{1\}, \{\mathbb{R}\}, \{\emptyset\}, \{1, \mathbb{R}\}, \{1, \emptyset\}, \{\mathbb{R}, \emptyset\}, \{1, \mathbb{R}, \emptyset\}\}$

4. (15 points) What is the cardinality of each of the following sets?
   i. $B = \emptyset$
   $|B| = 0$
   ii. $C = \{a, b, \{a\}, \{a, b\}, \emptyset, \{\emptyset\}, \mathbb{R}, \{\emptyset, \mathbb{R}\}, \mathbb{Z}\}$
   $|C| = 9$
   iii. $P(A)$ for $A = \{1, 2, 3, a, b, c, \square, \triangle, \Diamond, \alpha, \beta, \gamma\}$
   For finite sets A, $|P(A)| = 2^{|A|} = 2^{12} = 4096$

5. (10 points) Let A and B be sets. Complete the definitions.
   i. The set $A - B$ is the collection of all elements x such that $x \in A$ but $x \notin B$.
   ii. Set A is a **proper subset** of set B if for every $x \in A, x \in B$ but $A \neq B$.

6. (25 points) Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 4, 5, 8, 9\}$ and $B = \{2, 4, 5, 6, 9, 10\}$. List the members of the following sets.
   i. $A \cup B = \{1, 2, 4, 5, 6, 8, 9, 10\}$
   ii. $A \cap B = \{4, 5, 9\}$
   iii. $\overline{\bar{A} \cup B}$
   $\bar{A} \cup B = \{2, 3, 4, 5, 6, 7, 9, 10\}$ so $\overline{\bar{A} \cup B} = \{1, 8\}$
   iv. $A - B = \{1, 8\}$
   v. $A \oplus B = \{1, 2, 6, 8, 10\}$

7. i. (5 points) Let S be some mathematical statement. Describe the strategy to show that S is a true statement using the proof technique of *contradiction*.
   The technique of contradiction will assume that the statement S is false. One then proceeds with logical statements and deductions that result in the absurd or a contradiction of known facts. Thus, it cannot be the case S is false. If S cannot be false then S must be true.
   ii. (15 points) Prove $\sqrt{23}$ is irrational.
   Assume $\sqrt{23}$ is rational. This allows us to write $\sqrt{23} = \frac{a}{b}$ where $a, b \in \mathbb{Z}, b \neq 0$. Furthermore we can assume that a and b share no factors since we can always just reduce the fraction to its simplest terms. Now, if $\sqrt{23} = \frac{a}{b}$ then $23 = \frac{a^2}{b^2}$ and $a^2 = 23b^2$. So 23 divides a^2. Since 23 is prime we know that 23 must also divide a and we can write $a = 23k$ for some $k \in \mathbb{Z}$. Now we rewrite $a^2 = 23b^2$ as $(23k)^2 = 23b^2$ and see that $b^2 = \frac{(23k)^2}{23} = 23k^2$. So 23 divides b^2. Once again, since 23 is prime we know that 23 must also divide b. This, however, is a contradiction since a and b share no common factors. So, the assumption that $\sqrt{23}$ is rational is false and we've show that $\sqrt{23}$ is irrational.

8. Let A and B be sets.
   i. (5 points) Describe the strategy of the proof technique one uses to show that $A = B$.
   First show $A \subseteq B$. Second show $B \subseteq A$. Put the two together and $A = B$.
   ii. (10 points) Prove $A \oplus S = \overline{A}$ where S is the universal set.

First show $A \oplus S \subseteq \overline{A}$. Let $x \in A \oplus S$. This means that either $x \in A$ and $x \notin S$ or $x \notin A$ and $x \in S$. Since S is the universal set, x must be in S. Thus, $x \in A$ and $x \notin S$ cannot be true and it must be that $x \notin A$ and $x \in S$. Thus, $x \notin A$ and $x \in \overline{A}$.
Second show $\overline{A} \subseteq A \oplus S$. Let $x \in \overline{A}$. By definition of the universal set, $x \in S$. Thus $x \notin A$ and $x \in S$ which shows that $x \in A \oplus S$.