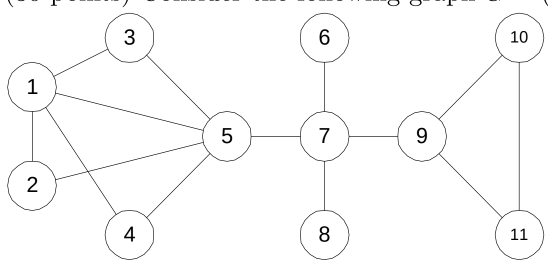
Math 4322 Quiz I
DeMaio Spring 2009

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. Where appropriate, simplify all answers to a single decimal expansion.

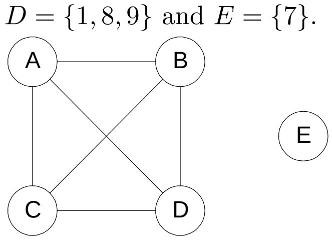
1. (15 points) Provide an algebraic proof that $\binom{n+1}{2}^2 - \binom{n}{2}^2 = n^3$.
   $\binom{n+1}{2}^2 - \binom{n}{2}^2 = \left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{n(n-1)}{2}\right)^2 = \frac{n^2((n+1)^2-(n-1)^2)}{4} = \frac{n^2\left((n^2+2n+1)-(n^2-2n+1)\right)}{4} = \frac{n^2(4n)}{4} = n^3$

2. (30 points) Consider the following graph $G = (V, E)$.

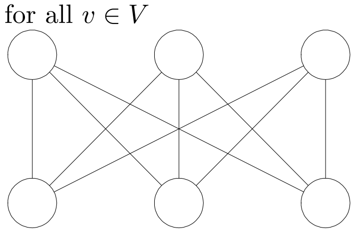


   i. Vertex 1 **is not** adjacent to vertex 7.
   ii. The $5-7$ edge **is** adjacent to the $7-9$ edge.
   iii. $N(7) = \{5, 6, 8, 9\}$
   iv. $\deg(9) = 3$
   v. The set of all pendants of G is $\{6, 8\}$
   vi. The set of all isolated vertices of G is $\emptyset$

3. (10 points) Draw the intersection graph for sets $A = \{1, 4, 5, 8, 9\}$, $B = \{2, 4, 5, 6, 9, 10\}$, $C = \{1, 2, 3\}$, $D = \{1, 8, 9\}$ and $E = \{7\}$.



4. (10 points) Construct a graph $G = (V, E)$ with $n = 6$ vertices and $e = 9$ edges such that $\deg(v) \leq 3$ for all $v \in V$



5. (10 points) Suppose a graph has 150 edges, 30 vertices of degree 4, and all others of degree 3. How many vertices does the graph have?
   By the Handshaking Lemma we know $\sum_{v \in V} \deg(v) = 2e$. We quickly see that $2e = 2 * 150 = 300$. Let x be the number of vertices of degree 3. Thus, $\sum_{v \in V} \deg(v) = 4 * 30 + 3x$. Setting $300 = 120 + 3x$ and solving for x yields $x = 60$. The graph has $30 + 60 = 90$ vertices.

6. (10 points) $\frac{504!}{502!} = \frac{504*503*502!}{502!} = 504 * 503 = 253\,512$

7. (15 points) Use induction to prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

I. Show that $S(1)$ is true. L.H.S. $\sum_{i=1}^{1} i = 1$. R.H.S. $\frac{1(1+1)}{2} = \frac{2}{2} = 1$. Thus, $S(1)$ is true.

II. Assume $S(n)$ is true: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and show $S(n+1)$ is true: $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$..

$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1)$ which by the inductive hypothesis is $\frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$.

8. (15 points) Six men and seven women stand in a line at the bookstore.

(a) How many arrangements of these people are possible?$13! = 6227\,020\,800$

(b) How many arrangements of these people are possible if the men stand in succession?$6! * 8! = 29\,030\,400$

(c) How many arrangements of these people are possible if the men and women must alternate positions within the line?$7! * 6! = 3628\,800$