Math 4322 Quiz II
DeMaio Spring 2010

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. Where appropriate, simplify all answers to a single decimal expansion.

1. (15 points) Prove that the 3×6 chessboard does not admit a closed knight's tour.

1	4	7	10	13	16
2	5	8	11	14	17
3	6	9	12	15	18

Vertices 2 and 14 have degree 2 and immediately force the closed cycle $2 - 7 - 14 - 9 - 2$ with no opportunity to include the other 14 vertices. Thus, no closed knight's tour exists.

2. (10 points) Find an open knight's tour on the 3×4 chessboard that begins at 1 and ends at 2.

1	4	7	10
2	5	8	11
3	6	9	12

Based on vertices of degree 2 (not counting 1 and 2) we must include the paths $1 - 8 - 3 - 4 - 11 - 6$ and $7 - 12 - 5 - 10 - 9$. If we now include the $6 - 7$ and $9 - 2$ edges we get the open tour $1 - 8 - 3 - 4 - 11 - 6 - 7 - 12 - 5 - 10 - 9 - 2$ which begins at 1 and ends at 2.

3. (5 points) With a brief proof explain why an open tour on the 3×100 chessboard that begins at 1 and ends at 3 cannot exist.

1	4	7	...	298
2	5	8	...	299
3	6	9	...	300

There exist 150 white squares and 150 black squares which the knight alternates between. However, 1 and 3 are both black and an alternating sequence of all black and white squares could not start and end on squares of the same color.

4. (10 points) Prove no closed knight's tour exists on the $4 \times n$ chessboard. No, you are not allowed to reference Schwenk's Theorem!
Label the squares of the board as below.

1	4	1	4	1	4	
2	3	2	3	2	3	...
3	2	3	2	3	2	
4	1	4	1	4	1	

Note that there are exactly n of each label. Furthermore, note that the only knight's move to and from a square with label 1 is via a square with label 2. Since there are an equal number of 1's and 2's there is no way to include the 3's and 4's and no Hamiltonian cycle exists.

5. (15 points) With a brief proof, show no Eulerian circuit exists on the $4 \times n$ chessboard for $n \geq 3$.

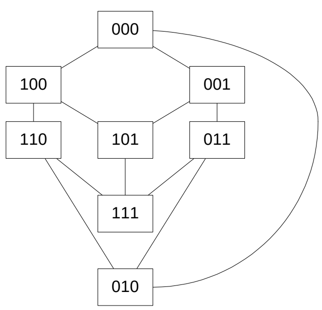
1	5	9	...	4n-3
2	6	10	...	4n-2
3	7	11	...	4n-1
4	8	12		4n

In an Eulerian graph, the degree of every vertex is even. Note however, that square 3 has degree 3.

6. (15 points) Prove that Q_n is Hamiltonian for all $n \geq 2$.
   We proceed by inducting on n. Since $Q_2 \cong C_4$ and every cycle graph is Hamiltonian it is clear that $S(2)$ is true. Now assume that Q_n is Hamiltonian and show that Q_{n+1} is Hamiltonian. Take two copies of the Hamiltonian cycle on Q_n. On the first copy append a 0 to every vertex label while on the second copy append a 1 to every label. We now have all the vertices of Q_{n+1}. Take any $a - b$ edge in the first copy of Q_n. In Q_{n+1} we have two occurrences of this edge:once as $a0 - b0$ and again as $a1 - b1$. Delete the $a0 - b0$ and $a1 - b1$ edges. Note that $a0$ and $a1$ differ only in the last bit and are adjacent in Q_{n+1}. Ditto for $b0$ and $b1$. We can create the $a0 - a1$ and $b0 - b1$ edges. We now have a Hamiltonian cycle in Q_{n+1}.

7. (10 points) State Kuratowski's Theorem.
   A graph G is planar if and only if it does not contain a subgraph homeomorphic to $K_{3,3}$ or K_5.

8. (10 points) Prove that Q_3 is planar.



A Planar Representation of Q_3

9. (20 points) Give examples of graphs $G = (V, E)$ such that G is

i. Hamiltonian and Eulerian; K_5
ii. Hamiltonian but not Eulerian; K_6
iii. Eulerian but not Hamiltonian; $K_{2,4}$
iv. neither Eulerian nor Hamiltonian. P_{10}