Math 3322 Quiz III
DeMaio Falls 2010

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (10 points) How many bit strings of length 8 exist? $2^8 = 256$
   How many bit strings of length 8 exist that begin with 1 and end with 11? $2^5 = 32$

2. (5 points) A particular brand of shirt comes in 12 colors, has a male and female version and comes in three sizes for each sex. How many different types of this shirt are made? $12 * 2 * 3 = 72$.

3. (10 points) How many different ways can you arrange the letters in the word
   i. game; $4! = 24$
   ii. baseball? $\frac{8!}{2!2!2!} = 5040$

4. (20 points) In the game of Clue, there are six suspects (Col. Mustard, Prof. Plum, Mr. Green, Mrs. Peacock, Miss Scarlet and Mrs. White), six possible weapons (Knife, Candlestick, Revolver, Rope, Lead Pipe and Wrench) and nine locations (Hall, Lounge, Dining Room, Kitchen, Ball Room, Conservatory, Billiard Room, Library and Study). The murder of Mr. Boddy was committed by one suspect, with one weapon in one location.
   i. How many different possible ways could the murder have been committed? $6 * 6 * 9 = 324$
   ii. How many ways could Miss Scarlet have committed the murder? $1 * 6 * 9 = 54$
   iii. How many different possible ways could the murder have been committed by Col. Mustard or Mrs. Peacock? These are disjoint cases. So, $1 * 6 * 9 + 1 * 6 * 9 = 108$.
   iv. How many different possible ways could the murder have been committed by Mr. Green and using the knife? $1 * 1 * 9 = 9$
   v. How many different possible ways could the murder have been committed by Mrs. White or using the rope? These are not disjoint cases. So, $1 * 6 * 9 + 6 * 1 * 9 - 1 * 1 * 9 = 99$.

5. (15 points) At a particular restaurant there are 5 meat options and 6 veggie options that are available as toppings on pizza. No doubles of any single topping are allowed.
   i. How many different 2 item pizzas exist? $\binom{11}{2} = 55$
   ii.. How many different 2 item pizzas exist with one meat and one veggie? $5 * 6 = 30$
   iii. How many different pizzas exist with at least three toppings? $2^{11} - \binom{11}{0} - \binom{11}{1} - \binom{11}{2} = 1981$

6. (15 points) Let $D = \{1, 2, 3, 4, 5\}$ and $R = \{a, b, c, d\}$.
   i. How many functions $f : D \to R$ exist? $4^5 = 1024$
   ii. How many one-to-one functions $f : D \to R$ exist? 0
   iii. How many onto functions $f : D \to R$ exist? $\binom{5}{2}4! = 240$

7. (10 points) How many positive integers not exceeding 1000 are divisible by either 6 or 8? $\left\lfloor \frac{1000}{6} \right\rfloor + \left\lfloor \frac{1000}{8} \right\rfloor - \left\lfloor \frac{1000}{\text{lcm}(6,8)} \right\rfloor = 250$

8. (10 points) State both the Pigeonhole Principle and the generalized Pigeonhole Principle.
   Pigeonhole Principle: If $k + 1$ pigeons are placed into k pigeonholes then at least one pigeonhole contains at least two pigeons.
   Generalized Pigeonhole Principle: If n pigeons are placed into k pigeonholes then at least one pigeonhole contains at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons.

9. (10 points) A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.
   i. How many balls must she select to be certain that she has at least 3 balls of the same color? Five since $\left\lceil \frac{4}{2} \right\rceil = 2$ and $\left\lceil \frac{5}{2} \right\rceil = 3$.
   ii. How many balls must she select to be certain that she has at least 3 red balls? 13

10. (10 points) Combinatorially prove $\binom{2n}{2} = 2\binom{n}{2} + n^2$ for $n \in \mathbb{Z}^+$.
Let $A = \{1, 2, ..., n, n+1, ..., 2n\}$ and S is the collection of all subsets of size 2 of A. On the one hand, $|S| = \binom{2n}{2}$. On the other hand partition A into $B = \{1, 2, ..., n\}$ and $C = \{n+1, ..., 2n\}$. How can we select two elements of A relative to B and C? We can select both from B in $\binom{n}{2}$ ways or both from C, again in $\binom{n}{2}$ ways. Or we could select one from each of B and C in n^2 ways. Thus, $|S| = 2\binom{n}{2} + n^2$. We have counted the same set S in two different ways and $\binom{2n}{2} = 2\binom{n}{2} + n^2$.