Math 4322 Quiz IV
DeMaio Spring 2009

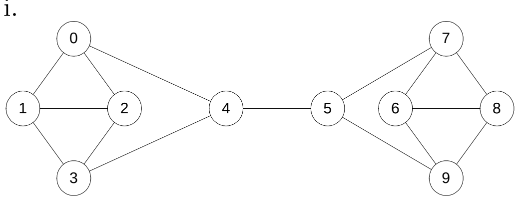
Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. Where appropriate, simplify all answers to a single decimal expansion.

1. (10 points) Let $G = (V, E)$ be a graph. Carefully define a *legal coloring* of a graph and the *chromatic number* of a graph.
A legal coloring of a graph G is an assignment of colors to every vertex such that adjacent vertices are assigned different colors. The chromatic number of a graph G is the minimum number of colors needed to legally color G.

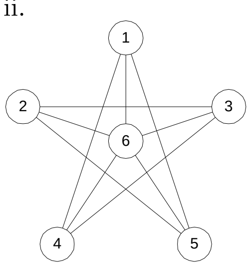
2. (20 points) Compute each of the following.
i. $\chi(K_n) = n$
ii. $\chi(K_{n,m}) = 2$
iii. $\chi\left(\overline{K_{n,m}}\right) = \max\{n, m\}$
iv. $\chi(C_n) = \left\{ \begin{array}{c} 2 \text{ if } n \text{ is even} \\ 3 \text{ if } n \text{ is odd} \end{array} \right\}$
v. $\chi(W_n) = \left\{ \begin{array}{c} 3 \text{ if } n \text{ is even} \\ 4 \text{ if } n \text{ is odd} \end{array} \right\}$

3. (20 points) Find and prove the correctness of the chromatic number of the following graphs.
i.



color	vertices
red	$\{0, 3, 5, 6\}$
blue	$\{2, 4, 8\}$
yellow	$\{1, 7, 9\}$

is a legal coloring of G with 3 colors. Thus, $\chi(G) \leq 3$. The existence of the triangle formed by the vertices $0, 1$ and 2 forces the use of at least 3 colors and $\chi(G) \geq 3$. Put these two facts together and we have shown that $\chi(G) = 3$.
ii.

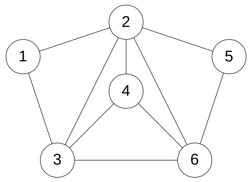


color	vertices
red	$\{1, 3\}$
blue	$\{2, 4\}$
yellow	$\{5\}$
green	$\{6\}$

is a legal coloring of H with 4 colors. Thus, $\chi(H) \leq 4$. The existence of the cycle of odd length formed by vertices 1, 5, 2, 3 and 4 forces the use of at least 3 colors. However, vertex 6 is adjacent to all five vertices in the cycle and a fourth color is required. Thus, $\chi(H) \geq 4$. Put these two facts together and we have shown that $\chi(H) = 4$.

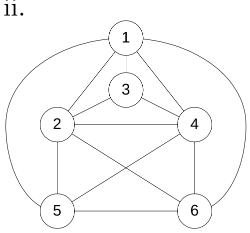
4. (10 points) Find $P(P_n, x)$.
$P(P_n, x) = x(x-1)^{n-1}$

5. (20 points) Determine the chromatic polynomial for each of the following graphs.
   i.



   $P(G,x) = x(x-1)(x-2)^3(x-3) = x^6 - 10x^5 + 39x^4 - 74x^3 + 68x^2 - 24x$

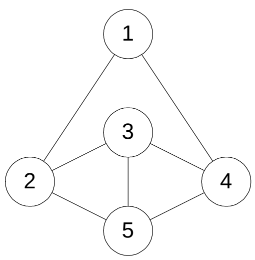
   ii.



   $P(H,x) = x(x-1)(x-2)(x-3)^2(x-4) = x^6 - 13x^5 + 65x^4 - 155x^3 + 174x^2 - 72x$

6. (10 points) For each of the graphs in the previous problem, find the number of different legal colorings using at most three colors.
   i. $P(G,0) = 0$
   ii. $P(H,0) = 0$

7. (10 points) Using the chromatic polynomial reduction formula, determine the chromatic polynomial of the following graph.



   $P(G,x) = x(x-1)^2(x-2)^2 - x(x-1)(x-2)(x-3) = x^5 - 7x^4 + 19x^3 - 23x^2 + 10x.$

8. (15 points) Let $G = (V,E)$ be a graph. Carefully define a *legal edge coloring* of a graph and the *edge chromatic number* of a graph. Find the edge chromatic number of W_n.
   A legal edge coloring of a graph G is an assignment of colors to every edge such that adjacent edges are assigned different colors. The edge chromatic number of a graph G is the minimum number of colors needed to legally edge color G. The edge chromatic number of W_n is n. It is clear that at least n colors are needed to color the edges since the center vertex has degree n. The remaining cycle of edges can now easily be colored with n colors.