Math 3322 Quiz IV
DeMaio Fall 2009

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (40 points) In the questions below suppose that a "word" is any string of five letters of the alphabet, with repeated letters allowed.
   i. How many words exist? $26^5 = 11\,881\,376$
   ii. How many words end with the letter B? $26^4 = 456\,976$
   iii. How many words begin with A and end with B? $26^3 = 17\,576$
   iv. How many words begin with A or B? $2 * 26^4 = 913\,952$
   v. How many words begin with A or end with B? $26^4 + 26^4 - 26^3 = 896\,376$
   vi. How many words have no vowels? $21^5 = 4084\,101$
   vii. How many words have exactly one vowel? $5 * 5 * 21^4 = 4862\,025$
   viii. How many words have at least one vowel? $26^5 - 21^5 = 7797\,275$

2. (5 points) How many functions are there from a set with 7 elements to a set with 10 elements? $10^7 = 10\,000\,000$

3. (5 points) How many one-to-one functions are there from a set with 7 elements to a set with 10 elements?$10 * 9 * 8 * 7 * 6 * 5 * 4 = \frac{10!}{3!} = 604\,800$

4. (10 points) Consider the twenty person club consisting of eight men and twelve women. How many ways can a President and Vice-President of the **same** gender be elected? $8 * 7 + 12 * 11 = 188$

5. (10 points) A **palindromic number** (or **palindrome**) is a number whose digits are the same when read left to right or right to left. So, 76,567 and 111 and 3,443 are all palidromic numbers while 123,123 and 78,871 are not. How many positive integer palindromes exist with six or seven digits? Be careful not to begin a number with 0! $9 * 10 * 10 * 1 * 1 * 1 + 9 * 10 * 10 * 10 * 1 * 1 * 1 = 9900$

6. (20 points) How many positive integers not exceeding 1,000 are divisible by 10 or 12? Note that $\text{lcm}(10, 12) = 60$ There are $\left\lfloor \frac{1000}{10} \right\rfloor + \left\lfloor \frac{1000}{12} \right\rfloor - \left\lfloor \frac{1000}{60} \right\rfloor = 167$ positive integers not exceeding 1,000 are divisible by 10 or 12.

7. (10 points) In the game of *Clue*, there are six suspects, six possible weapons and nine locations. The murder of Mr. Boddy was committed by one suspect, with one weapon in one location. How many ways can the murder have been committed;
   i. by Col. Mustard or Mrs. Peacock; $2 * 6 * 9 = 108$
   ii. in the study or using the candlestick? $6 * 6 * 1 + 6 * 1 * 9 - 6 * 1 * 1 = 84$

8. (20 points) There are n chairs and some collection of people (including none) will sit in the seats but there will always be at least one empty chair between any two people. Let A_n be the number of antisocial ways to seat some number of people in these n seats as described. Construct all possible arrangements and compute A_n for all values up to $n = 4$. Find and prove the correctness of a recursive formula for A_n. Let 0 represent an empty seat and let 1 represent a seat with a person.

n	arrangements	A_n
1	0,1	2
2	00,10,01	3
3	000,101,100,010,001	5
4	0000,1000,0100,0010,0001,1001,1010,0101	8

We conjecture that $A_n = f_{n+2}$. Certainly the first few values are the same. It remains to show that A_n follows the same recurrence. Show $A_n = A_{n-1} + A_{n-2}$. We can partition all A_n arrangements into those that end with an empty seat and those that end with a person. An empty seat can be added to any arrangement of $n - 1$ seats. Thus, there are A_{n-1} arrangements of n seats that end with an empty seat. An empty seat and a person can be added to the end of every arrangement of $n - 2$ seats. Thus, there exist A_{n-2} arrangements that end with a person. Hence, $A_n = A_{n-1} + A_{n-2}$.