Math 3322 Quiz V
DeMaio Fall 2009

Name______________________________

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (10 points) The Jones family is going to fill up their cooler to its twelve can capacity with sodas from a machine. Soda choices are Coke, Diet Coke, Cherry Coke, Caffeine Free Coke, Caffeine Free Diet Coke, Sprite, Ginger Ale, and Surge. How many different ways can the Jones family cooler be filled with drinks?
   This is an unordered selection with repetition where $n = 8$ and $k = 12$. So, $\binom{8+12-1}{12} = \binom{19}{12} = 50\,388$ combinations exist.

2. (25 points) i. A store sells 5 kinds of cookies. How many ways can John buy 10 cookies?This is an unordered selection with repetition where $n = 5$ and $k = 10$. So, $\binom{5+10-1}{10} = \binom{14}{10} = 1001$ combinations exist.
   ii. Consider the equation $x+y+z = 10$. If x, y and z are all non-negative integers, how many different solutions exist? Keep in mind that the solution $x = 3$, $y = 4$ and $z = 3$ is different from the solution $x = 3$, $y = 3$ and $z = 4$.
   This is an unordered selection with repetition where $n = 3$ and $k = 10$. So, $\binom{3+10-1}{10} = \binom{12}{10} = 66$ combinations exist.
   iii. A store sells 5 kinds of cookies. How many ways can John buy 10 cookies if he must buy at least one of each type of cookie?
   This is an unordered selection with repetition where we go ahead and place one of each cookie in the bag to purchase. Now, $n = 5$ and $k = 5$. So, $\binom{5+5-1}{5} = \binom{9}{5} = 126$ combinations exist.
   iv. Consider the equation $x + y + z = 10$. If x, y and z are all positive integers, how many different solutions exist? Keep in mind that the solution $x = 3$, $y = 4$ and $z = 3$ is different from the solution $x = 3$, $y = 3$ and $z = 4$.
   This is an unordered selection with repetition where we go ahead and place one unit into each variable to avoid 0 in our solution set. Now, $n = 3$ and $k = 7$. So, $\binom{3+7-1}{7} = \binom{9}{7} = 36$ combinations exist.
   v. Why has Professor DeMaio grouped these seemingly unrelated questions together?
   They may appear to be completely different settings, cookies vs. integer solutions to an equation, but they are both just problems from the *unordered selections with repetition* category

3. (30 points) Susan buys an economy pack of fifty pens. The pens are identical except for color. There are ten of each of five different colors.
   i. How many different ways can Susan select four pens of different colors to take to work? This is an unordered selection without repetition and can be done in $\binom{5}{4} = 5$ ways.
   ii. How many different ways can Susan select ten pens to take to work? This is an unordered selection with repetition where $n = 5$ and $k = 10$. So, $\binom{5+10-1}{10} = \binom{14}{10} = 1001$ combinations exist.
   iii. How many different ways can Susan select six pens to take to work with at least two different colors in the mix? $\binom{5+6-1}{6} - 5 = 204$

4. (20 points) Provide a combinatorial proof that $\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn$ for $m, n \in \mathbb{Z}^+$.
   Let $A = \{1, 2, 3, ..., m-1, m, m+1, ..., m+n\}$ and let S be the collection of all two element subsets of A. One the one hand, $|S| = \binom{m+n}{2}$. On the other hand partition A into $B = \{1, 2, 3, ..., m\}$ and $C = \{m+1, m+2, ...m+n\}$. Note that $|B| = m$ while $|C| = n$. How can we select two elements of A relative to B and C? We can select two elements from B in $\binom{m}{2}$ ways or we can select two elements from C in $\binom{n}{2}$ or we can select one element from B and one element from C in mn ways. Thus, $|S| = \binom{m}{2} + \binom{n}{2} + mn$. This shows that $\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn$ for $m, n \in \mathbb{Z}^+$.

5. (20 points) Provide an algebraic proof that $\binom{n}{k}k = n\binom{n-1}{k-1}$. Remember you cannot work both sides of the equation down to a true statement to prove the original identity.
   $k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!} = n\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = n\binom{n-1}{k-1}$

6. (10 points) State the values of x and y that make the following identity true. **Think combinatorially!**

$$\binom{kn}{2} = x\binom{n}{2} + yn^2 \text{ for } k, n \in \mathbb{Z}^+$$

$x = k$ and $y = \binom{k}{2}$