Math 3322 Quiz III
DeMaio Fall 2008

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. Where appropriate, simplify all answers to a single decimal expansion.

1. (15 points) Let $g_0 = 1$. Let $g_n = 2^{g_{n-1}}$ for $n \geq 1$. Compute g_1, g_2, g_3 and g_4.
   $g_1 = 2^{g_0} = 2^1 = 2$
   $g_2 = 2^{g_1} = 2^2 = 4$
   $g_3 = 2^{g_2} = 2^4 = 16$
   $g_4 = 2^{g_3} = 2^{16} = 65536$
2. (10 points) Give a recursive definition of the set of positive integer powers of 5.
   Let $P_1 = 5$. Let $P_n = 5P_{n-1}$ for $n \geq 2$.
3. (10 points) State the recursive definition of the Fibonacci sequence.
   Let $f_0 = 0$ and $f_1 = 1$. For $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$.
4. (10 points) Complete the table of Fibonacci numbers.

n	0	1	2	3	4	5	6	7	8	9	10
f_n	0	1	1	2	3	5	8	13	21	34	55

5. (15 points) There are n chairs and some collection of people (including none) will sit in the seats but there will always be at least one empty chair between any two people. Let A_n be the number of antisocial ways to seat some number of people in these n seats as described. Construct all possible arrangements and compute A_n for all values up to $n = 3$. Find and prove the correctness of a formula for A_n.

   Let 0 represent an empty seat and let 1 represent a seat with a person.

n	arrangements	A_n
1	0,1	2
2	00,10,01	3
3	000,101,100,010,001	5

   We conjecture that $A_n = f_{n+2}$. Certainly the first few values are the same. It remains to show that A_n follows the same recurrence. Show $A_n = A_{n-1} + A_{n-2}$. We can partition all A_n arrangements into those that end with an empty seat and those that end with a person. An empty seat can be added to any arrangement of $n - 1$ seats. Thus, there are A_{n-1} arrangements of n seats that end with an empty seat. An empty seat and a person can be added to the end of every arrangement of $n - 2$ seats. Thus, there exist A_{n-2} arrangements that end with a person. Hence, $A_n = A_{n-1} + A_{n-2}$.

6. (15 points) Use induction to prove $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$ for the Fibonacci sequence and $n \in Z^+$.

   First show that the base case of $n = 1$ hold true.

R.H.S. $\sum_{i=1}^{1} f_i^2 = f_1^2 = 1^2 = 1$

L.H.S. $f_1 f_{1+1} = f_1 f_2 = 1 * 1 = 1$

Assume the statement is true for n : $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$. Show that the statement is true for $n+1$: $\sum_{i=1}^{n+1} f_i^2 = f_{n+1} f_{n+2}$.

Note that $\sum_{i=1}^{n+1} f_i^2 = \sum_{i=1}^{n} f_i^2 + f_{n+1}^2$ which by the inductive hypothesis is $f_n f_{n+1} + f_{n+1}^2 = f_{n+1}(f_n + f_{n+1}) = f_{n+1} f_{n+2}$.

7. (5 points) How many bit strings of length 6 exist? $2^6 = 64$

8. (5 points) How many bit strings of length 6 exist that end and begin with 0?$2^4 = 16$

9. (5 points) Consider a twenty person club. How many different ways can a President, Vice-President and Treasurer be elected? $20 * 19 * 18 = 6840$

10. (5 points) Consider the twenty person club made up of eight men and twelve women. How many ways can a President and Vice-President of opposite gender must be selected? $12 * 8 * 2 = 192$

11. (5 points) A theater concession counter offers four different sizes of drinks and eight different choices of beverages. How many different ways can a drink be ordered?$4 * 8 = 32$

12. (5 points each) Two married couples, two single men and one single woman sit in a row of seven consecutive seats. How many ways can they be seated

i. with no restrictions? $7! = 5040$

ii. alternating genders? With 4 men and 3 women the only way to alternate gender is with a MWMWMWM arrangement. Hence there exist $4 * 3 * 3 * 2 * 2 * 1 * 1 = 4! * 3! = 144$ different arrangements.

iii. such that the women are all consecutive? There are 3! ways to arrange the women in a consecutive block of seats. Now we have 4 men and 1 block of women to arrange. There are a total of $3! * 5! = 720$ legal arrangements.

iv. such that spouses sit next to one another. As in part iii we need to arrange each married couple in a block. Each married couple can be arranged in $2! = 2$ ways. Now we are left with 5 types of objects to arrange in seats. There are a total of $2! * 2! * 5! = 480$ different arrangements.