Math 4322 Test I
DeMaio Spring 2010

Name____

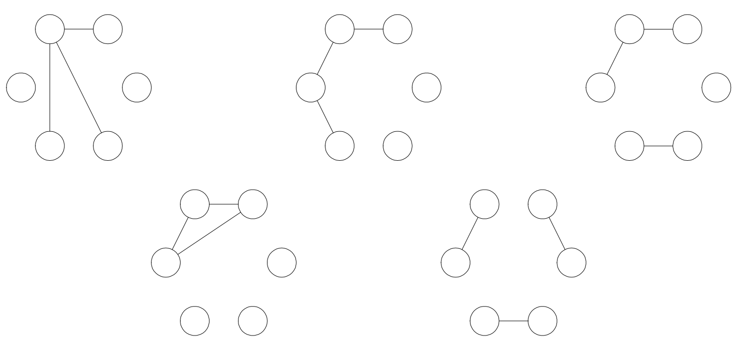
Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. Where appropriate, simplify all answers to a single decimal expansion.

1. (15 points) i. A graph $G = (V, E)$ is regular if $\deg(v) = k$ for all $v \in V$ for some fixed integer k.
   A vertex $v \in V$ is a pendant if $\deg(v) = 1$.
   ii. Can a pendant exist in a regular graph with $n = 11$ vertices? If yes, draw such a graph. If no, prove why. No! If G is regular and contains a pendant then every vertex is a pendant. This forces $\sum_{v \in V} \deg(v) = \sum_{v \in V} 1 = 11$ which odd. However, by the Handshaking lemma $\sum_{v \in V} \deg(v) = 2e$ which must be even.

2. (5 points) Let $G = (V, E)$ be a simple graph with n vertices and e edges. How many edges exist in $\overline{G}$? $\binom{n}{2} - e$
   (5 points) Find a self-complementary simple graph with $n = 4$ vertices. P_4.
   (5 points) Prove you cannot find a self-complementary simple graph with $n = 6$ vertices. If G is self-complementary then G and $\overline{G}$ have the same number of edges. Thus $\binom{n}{2} = 2e$ which is even. However, $\binom{6}{2} = 15$ which is odd. No graph with six vertices can be self-complementary

3. (10 points) Compute the number of unequal labeled graphs that exist with
   i. $n = 6$ vertices and $e = 7$ edges. $\binom{\binom{6}{2}}{7} = 6435$
   ii. $n = 5$ vertices. $2^{\binom{5}{2}} = 1024$

4. (10 points) Construct all non-isomorphic graphs with $n = 6$ vertices and $e = 3$ edges.

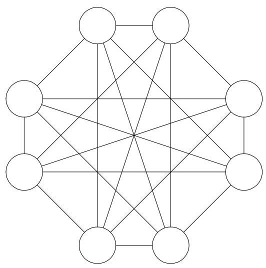
   The five non-isomorphic graphs on six vertices and three edges

5. (10 points) Let $G = (V, E)$ be a graph with $n \geq 2$ vertices. Prove there exist $x, y \in V$ such that $\deg(x) = \deg(y)$.
   Note that for every vertex $v \in V$, $0 \leq \deg(v) \leq n - 1$. So, n different vertex degrees exist. However, the degrees 0 and $n - 1$ cannot coexist in a graph since that would imply one vertex is adjacent to all other vertices in the graph, including an isolated vertex. Thus, we place n vertices into $n - 1$ pigeonholes labeled by a vertex degree and, according to the pigeonhole principle, at least two vertices have been placed into the same pigeonhole. These vertices are the $x, y \in V$ such that $\deg(x) = \deg(y)$.

6. (10 points) Prove $\overline{C_{2n}} \not\cong K_{n,n}$ for $n = 3$. The graph $\overline{C_6}$ is not bipartite (vertices 1,3 and 5 form a triangle) while $K_{3,3}$ is bipartite.

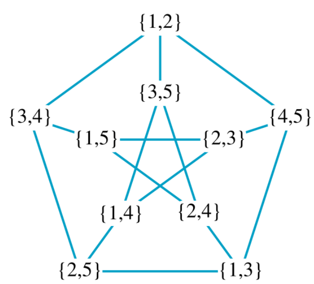
7. (10 points) Prove $\overline{C_{2n}} \not\cong K_{n,n}$ for $n \geq 4$. The graph $K_{n,n}$ is regular of degree n. The graph $\overline{C_{2n}}$ is regular of degree $2n - 3$. When $n \geq 4$, $2n - 3 > n$ and the two graphs have different degree sequences.

8. (10 points) Determine with proof when the sequence of n 5's is graphical. If the sequence contains an odd number of entries then the graph fails to exist since the sum of the degrees is not even. If the graph consists of an even number of entries then we are left with three possibilities: the remainder after division by 6 is 0, 2 or 4. If 6 divides n then the sequence can be realized with $\frac{n}{6}$ copies of K_5. If 6 does not divide n and leaves a remainder of 4 then the sequence can be realized by a single $K_{5,5}$ and $\frac{n-10}{6}$ copies of K_5. If the remainder after division by 6 is 2 then take a single copy of G (as shown below) and $\frac{n-8}{6}$ copies of K_5.



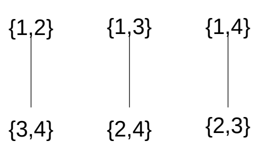
A regular graph G of degree 5 with 8 vertices

9. (25 points) The Kneser graph $K(n,k)$ is the graph whose vertices correspond to the k-element subsets of a set of n elements, and where two vertices are adjacent if and only if the two corresponding sets are disjoint.



$K(5,2)$

i. Draw $K(4,2)$.



$K(4,2)$

ii. The Kneser graph $K(n,k)$ is a regular graph. Compute the degree. Explain. Every set S of k elements will be adjacent to all other disjoint sets of k elements. There are $n-k$ elements not in S. Thus, there are $\binom{n-k}{k}$ k element sets that are disjoint from S.
iii. Describe $K(n,1)$. $K(n,1) \cong K_n$.
iv. Describe $K(100,51)$. $K(100,51) \cong N_{\binom{100}{51}}$.
v. Generalize part iv to a theorem.
The Kneser graph $K(n,k) \cong N_{\binom{n}{k}}$ when $k > \frac{n}{2}$.