Math 3322 Test I
DeMaio Fall 2011

Name______________________________

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (35 points) Compute the following.
   i. $\left\lceil \left(\frac{1}{33}\right)^{250} \right\rceil = 1$
   ii. $\lceil \pi \rceil - \lfloor e \rfloor = 2$
   iii. $\frac{250!}{3!247!} = \frac{250*249*248}{6} = 2573\,000$
   iv. $\sum\limits_{i=1}^{350} i = 61\,425$
   v. $\sum\limits_{i=1}^{200} i^2 = 2686\,700$
   vi. $\sum\limits_{i=300}^{500} i = 80\,400$
   vii. $\prod\limits_{k=1}^{100} k^{\left\lfloor \frac{1}{k^2} \right\rfloor} = 1$

2. (5 points) True or False? $\lfloor x \lceil y \rceil \rfloor = xy$ for $x, y \in \mathbb{R}$. If true, prove it. If false, provide a counter example.
   False. Let $x = y = \frac{1}{2}$.
   Note that $\left\lfloor \frac{1}{2} \left\lceil \frac{1}{2} \right\rceil \right\rfloor = 0$ while $\frac{1}{2}^2 = \frac{1}{4}$.

3. (5 points) Find the first 8 terms of the sequence whose n^{th} term is the number of letters in the English word for the index n.
   One, two, three, four, five, six, seven, eight
   translates into
   $3, 3, 5, 4, 4, 3, 5, 5$

4. (5 points) Threaten every square on the 5×7 chessboard with 3 queens.
   One solution is to place queens on cells $(3, 1)$, $(3, 4)$ and $(3, 7)$.

5. (10 points) True or False? You can tile the 5×5 chessboard with 3 corners removed with dominoes. If true, prove it. If false, provide a counter example.
   False. If you remove 3 corners you are left with 10 black squares and 12 white squares. Since every domino covers exactly one white square and one black square it will be impossible to tile the board.

6. (10 points) Let f be the function that assigns to each bit string, three times the number of 0's in the bit string. State the domain and range of f.
   The domain is the collection of all bit strings. The range is all non-negative multiples of 3. In list form the range is $\{0, 3, 6, 9, 12, ...\}$.

7. (10 points) Give an example of a function $f : \mathbb{Z}^+ \to \mathbb{Z}^-$ that is neither one-to-one nor onto. Explain why your example is correct.
   Let $f(x) = -5$. The function f is not onto since -6 has no pre-image. The function f is not 1-1 since $f(1) = f(2) = -6$ but, obviously, $1 \neq 2$.

8. (10 points) Use induction to prove $\sum\limits_{k=1}^{n} k \times k! = (n+1)! - 1$ for all $n \in \mathbb{Z}^+$.
   $\sum\limits_{k=1}^{1} k \times k! = 1$ and $(1+1)! - 1 = 1$ so $S(1)$ is true. Assume $\sum\limits_{k=1}^{n} k \times k! = (n+1)! - 1$ and show $\sum\limits_{k=1}^{n+1} k \times k! = (n+2)! - 1$. $\sum\limits_{k=1}^{n+1} k \times k! = \sum\limits_{k=1}^{n} k \times k! + (n+1) \times (n+1)!$. By the inductive hypothesis this is $(n+1)! - 1 + (n+1) \times (n+1)! = (n+1)!\,[1 + (n+1)] - 1 = (n+2)! - 1$. Hence, $\sum\limits_{k=1}^{n} k \times k! = (n+1)! - 1$ for all $n \in \mathbb{Z}^+$.

9. (10 points) Use induction to prove that 6 divides $n^3 - n$ for all $n \in \mathbb{Z}^+$.
Since $\frac{1^3-1}{6} = 0 \in \mathbb{Z}$ then $S(1)$ is true. Now assume $\frac{n^3-n}{6} \in \mathbb{Z}$ and show that $\frac{(n+1)^3-(n+1)}{6} \in \mathbb{Z}$. So, $\frac{(n+1)^3-(n+1)}{6} = \frac{\left(n^3+3n^2+3n+1\right)-(n+1)}{6} = \frac{n^3-n}{6} + \frac{3n^2+3n}{6}$. Note that $\frac{n^3-n}{6} \in \mathbb{Z}$ by the inductive hypothesis. If we can show that $\frac{3n^2+3n}{6} \in \mathbb{Z}$ then we will be done. We quickly get $\frac{3n^2+3n}{6} = \frac{3n(n+1)}{3\times 2} = \frac{n(n+1)}{2}$. Either n or $n+1$ is even so $\frac{n(n+1)}{2} \in \mathbb{Z}$. Thus, $\frac{(n+1)^3-(n+1)}{6}$ is a sum of two integers which is an integer. Hence, 6 divides $n^3 - n$ for all $n \in \mathbb{Z}^+$.

10. (10 points) Let $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.
Compute $L_2, ..., L_{10.}$
$L_2 = 3$, $L_3 = 4$, $L_4 = 7$, $L_5 = 11$, $L_6 = 18$, $L_7 = 29$, $L_8 = 47$, $L_9 = 76$, and $L_{10} = 123$.