Math 3322 Test 1
DeMaio Spring 2010

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. Where appropriate, simplify all answers to a single decimal expansion.

Preliminaries

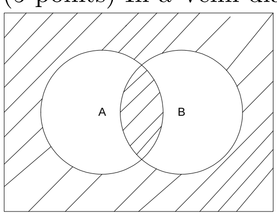
1. (5 points) List the members of the set $S = \left\{x | x \in Z^+, 50 \leq x^3 \leq 150\right\}$.
   $S = \{4, 5\}$

2. (5 points) Construct $P(A)$ for $A = \{*, a, 3\}$.
   $P(A) = \{\emptyset, \{*\}, \{a\}, \{3\}, \{*, a\}, \{*, 3\}, \{a, 3\}, \{*, a, 3\}\}$

3. (5 points) Compute $|P(A)|$ for $A = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31\}$.
   $|P(A)| = 2^{11} = 2048$

4. (5 points) Give an example of sets A and B such that B is a proper subset of A and $|A| = |B|$.
   Let $A = Z$ and let $B = Z^+$.

5. (5 points) In a Venn diagram, shade $\overline{A \oplus B}$.



6. (5 points) Let A be the set of students who live within one mile of campus. Let B be the set of all students who walk to class. What does it mean to say Mary is a member of the set $\overline{A} \cap B$?
   Mary walks more than one mile to her classes on campus.

7. (5 points) Compute $\left\lfloor -\frac{3}{2} + \left\lfloor \frac{1}{2} + \left\lceil -\frac{5}{2} \right\rceil \right\rfloor \right\rfloor$.
   $\left\lfloor -\frac{3}{2} + \left\lfloor \frac{1}{2} + \left\lceil -\frac{5}{2} \right\rceil \right\rfloor \right\rfloor = -4$

8. (5 points) Find the domain and range of the function that assigns to each positive integer its last digit.
   The domain is Z^+. The range is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

9. (5 points) Give an example of a function $f : \mathbb{Q} \to Z^+$ that is neither one-to-one nor onto.
   Let $f(\frac{a}{b}) = a^2 b^2$.

10. (5 points) Compute $\sum_{i=50}^{175} i$.
    $\sum_{i=50}^{175} i = \sum_{i=1}^{175} i - \sum_{i=1}^{49} i = \frac{175*176}{2} - \frac{49*50}{2} = 14\,175.$

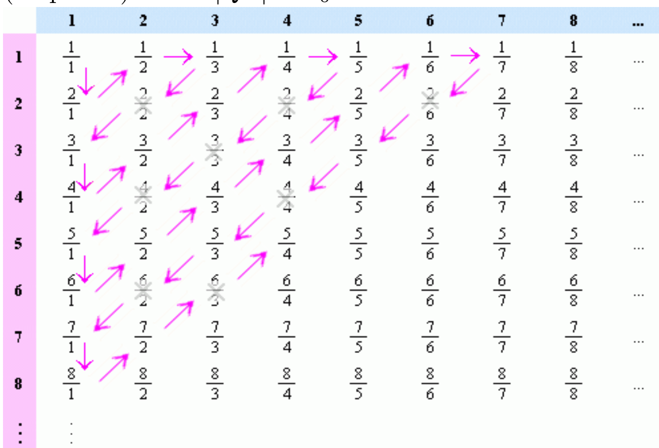
11. (5 points) Compute $\prod_{-533}^{278} \left(i^3 - 8\right)$.
    Since 2 is an integer between -533 and 278 the product contains the term $2^3 - 8 = 0$ and is 0.

12. (5 points) Compute $\frac{100!}{95!5!}$.
    $\frac{100!}{95!5!} = \frac{100*99*98*97*96}{120} = 75\,287\,520.$

Problems

13. (10 points) True or False? If true, prove it. If false, provide a counter-example.
    $(j + k)! = j! + k!$
    False! Let $k = j = 2$. Then $(j + k)! = (2 + 2)! = 41 = 24$ while $2! + 2! = 2 + 2 = 4$.

14. (15 points) Prove $|Q^+| = \aleph_0$.



from http://www.homeschoolmath.net/teaching/rational-numbers-countable.php.

15. (15 points) Use mathematical induction to prove $\sum\limits_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in Z^+$.

I. Show that $S(1)$ is true. L.H.S. $\sum\limits_{i=1}^{1} i^3 = 1^3 = 1$. R.H.S. $\frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$. Thus, $S(1)$ is true.

II. Assume $\sum\limits_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ and show $\sum\limits_{i=1}^{n+1} i^3 = \frac{(n+1)^2(n+2)^2}{4}$.
$\sum\limits_{i=1}^{n+1} i^3 = \sum\limits_{i=1}^{n} i^3 + (n+1)^3$ which by the inductive hypothesis is $\frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} = \frac{(n+1)^2\left[n^2+4(n+1)\right]}{4} = \frac{(n+1)^2\left[n^2+4n+4\right]}{4} = \frac{(n+1)^2(n+2)^2}{4}$.

16. (15 points) Let A and B be sets such that $|A| = |B| = \aleph_0$. Prove $|A \cup B| = \aleph_0$.
Since A and B are countable sets we can write each as $a_1, a_2, a_3...$ and $b_1, b_2, b_3....$ We can now write $A \cup B$ as $a_1, b_1, a_2, b_2, a_3, b_3, ...$ Since we can order the infinite number of elements of $A \cup B$ it follows that $|A \cup B| = \aleph_0$.