Math 4322 Test I
DeMaio Spring 2009

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. Where appropriate, simplify all answers to a single decimal expansion.

Preliminaries

1. (15 points) First, dominate a 5x7 board with 3 queens. Second, use that solution to then dominate a 6x8 board with 4 queens. You cannot receive credit for the second part of this problem if your solution does not build on the solution to the first part!

First:

Q			Q			Q

Second:

Q			Q			Q	
							Q

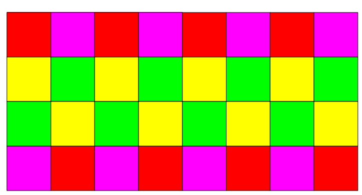
2. (10 points) Prove a closed knight's tour does not exist for the 3×6 board. No, you are not allowed to reference Schwenk's Theorem!

1	4	7	10	13	16
2	5	8	11	14	17
3	6	9	12	15	18

Note that squares 5 and 17 have degree 2. This immediately forces the closed cycle $5-10-17-12-5$. We are now unable to include other vertices in the knight's tour. The same argument works with vertices 2 and 14 forcing the closed cycle $2-7-14-9-2$.

3. (10 points) Prove a closed knight's tour does not exist for the $4 \times n$ board. No, you are not allowed to reference Schwenk's Theorem!

For $4 \times n$ boards we will use a different coloring of the board.



Coloring the $4 \times n$ Chessboard

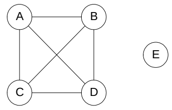
Note that a knight must move from a red square to a yellow square and a knight must move from a purple square to a green square. Two closed cycles are now forced and no closed tour exists for the $4 \times n$ board.

4. (5 points) Find an open knight's tour on the 3 by 4 board that begins at 1 and ends at 12.

1	2	3	4
5	6	7	8
9	10	11	12

$1-7-9-2-8-10-3-5-11-4-6-12$

5. (5 points) Draw the intersection graph for sets $A = \{1, 4, 5, 8, 9\}$, $B = \{2, 4, 5, 6, 9, 10\}$, $C = \{1, 2, 3\}$, $D = \{1, 8, 9\}$ and $E = \{7\}$.

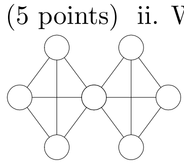


6. (15 points) i. Complete the definitions:
A graph $G = (V, E)$ is regular if $\deg(v) = k$ for all $v \in V$ for some fixed integer k.
A vertex $v \in V$ is a pendant if $\deg(v) = 1$.
ii. Can a pendant exist in a regular graph with $n = 11$ vertices? If yes, draw such a graph. If no, prove why.
No! If G is regular and contains a pendant then every vertex is a pendant. This forces $\sum_{v \in V} \deg(v) = \sum_{v \in V} 1 = 11$ which odd. However, by the Handshaking lemma $\sum_{v \in V} \deg(v) = 2e$ which must be even.

7. (15 points) For what values of n is W_n
i. bipartite; Never. Every W_n contains a triangle.
ii. Eulerian; Never. The degree of every vertex except the center vertex is three.
iii. Hamiltonian? Always for $n \geq 3$.

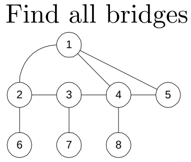
8. (5 points) Let $G = (V, E)$ be a graph with e edges. How many edges exist in $\overline{G}$?
$\binom{n}{2} - e$
(10 points) Find a self-complementary simple graph with $n = 5$ vertices. K_5
(5 points) Prove you cannot find a self-complementary simple graph with $n = 6$ vertices. If G is self-complementary then G and $\overline{G}$ have the same number of edges. Thus $\binom{n}{2} = 2e$ which is even. However, $\binom{6}{2} = 15$ which is odd. No graph with six vertices can be self-complementary.

9. (5 points) i. State Ore's Theorem.
Let $G = (V, E)$ be a graph. If $\deg(x) + \deg(y) \geq n$ for every pair of non-adjacent vertices $x, y \in V$ then G is Hamiltonian.
(5 points) ii. What does Ore's Theorem tell us about the following graph. Explain



Nothing! This graph does not satisfy the conditions of Ore's theorem. Since Ore's theorem is not an if and only if statement it cannot be used to show that a graph is not Hamiltonian.

10. (10 points) Provide the definitions of a bridge and a cutvertex in a graph $G = (V, E)$. A bridge is an edge whose removal increases the number of components in a graph. A cutvertex is a vertex whose removal increases the number of components in a graph.
Find all bridges and cutvertices in the following graph.



Bridges: $2 - 6, 3 - 7, 4 - 8$
Cutvertices: 2, 3 and 4