Math 3322 Test I
DeMaio Fall 2009

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (5 points) Let $A = \{1, 2, 3, 4\}$. True or False: $\emptyset \in A$? Explain.
   False. The empty set is a subset of every set but not necessarily a member of every set. In this case $\emptyset \notin A$.

2. (5 points) Give an example of sets A and B such that $A \subset B$ yet $|A| = |B|$.
   Many examples exist but only when A and B are both sets of infinite cardinality.
   Let $A = \mathbb{Z}^+$ and let $B = \mathbb{Z}$. Note that $\mathbb{Z}^+ \subset \mathbb{Z}$ and $|\mathbb{Z}^+| = |\mathbb{Z}| = \aleph_0$.

3. (10 points) Compute
   i. $\lfloor e^{\lceil \pi \rceil} \rfloor = \lfloor e^4 \rfloor = 54$
   ii. $\frac{879!}{876!} = \frac{879*878*877*876!}{876!} = 879 * 878 * 877 = 676\,835\,274$
   iii. $\prod_{i=-17}^{15} 2^i = \frac{1}{2^{16}*2^{17}} = \frac{1}{8589\,934\,592} = 1.164\,2 \times 10^{-10}$

4. (5 points) Let $S = \{1, 3, 5, 7\}$.
   $\sum_{j \in S} (j-1)^2 = 0^2 + 2^2 + 4^2 + 6^2 = 56$

5. (10 points) $\sum_{i=56}^{123} i^2 = \sum_{i=56}^{123} i^2 + \sum_{i=1}^{55} i^2 - \sum_{i=1}^{55} i^2 = \sum_{i=1}^{123} i^2 - \sum_{i=1}^{55} i^2 = 570\,894$

6. (5 points) Compute $7 + 14 + 21 + 28 + ... + 896 + 903$.
   $7 + 14 + 21 + 28 + ... + 896 + 903 = 7(1 + 2 + 3 + ... + 129) = 7 \sum_{i=1}^{129} i = 7\left(\frac{129*130}{2}\right) = 58\,695$

7. (10 points) Find the domain and range of the function that assigns to each bit string the number of ones in that string.
   The domain is the collection of all bit strings. The range is the non-negative integers.

8. (10 points) Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ via $f(n) = \lceil \frac{n}{2} \rceil$. Determine, with proof, if $f(n)$ is one-to-one and/or onto.
   The function $f(n) = \lceil \frac{n}{2} \rceil$ is not 1-1. This is evidently true since $f(1) = \lceil \frac{1}{2} \rceil = 1$, $f(2) = \lceil \frac{2}{2} \rceil = 1$, yet $1 \neq 2$.
   The function $f(n) = \lceil \frac{n}{2} \rceil$ is onto. Consider any integer $y \in \mathbb{Z}$. Since y is an integer, $2y$ is also an integer. Note that $f(2y) = \lceil \frac{2y}{2} \rceil = \lceil y \rceil = y$ and every integer has a pre-image.

9. (5 points) Find the terms a_0, a_1, a_2 and a_3 of the sequence $\{a_n\}$ where $a_n = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$.
   $a_0 = \lfloor \frac{0}{2} \rfloor + \lceil \frac{0}{2} \rceil = 0$ $\quad a_1 = \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil = 1$
   $a_2 = \lfloor \frac{2}{2} \rfloor + \lceil \frac{2}{2} \rceil = 2$ $\quad a_3 = \lfloor \frac{3}{2} \rfloor + \lceil \frac{3}{2} \rceil = 3$

10. (5 points) Find the next three terms of the sequence $3, 6, 11, 18, 27, 38, 51, 66, 83, 102$.
    The difference between terms is $3, 5, 7, 9, ...$
    $102 - 83 = 19$. So the next three terms are
    $102 + 21 = 123$;
    $123 + 23 = 146$;
    and $146 + 25 = 171$.

11. (10 points) i. Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. List the members of $A \times B$.
    $A \times B = \{(a, y), (a, z), (b, y), (b, z), (c, y), (c, z), (d, y), (d, z)\}$
    ii. Let $|A| = 35$ and $|B| = 27$. Determine $|A \times B|$.
    $|A \times B| = |A| * |B| = 35 * 27 = 945$

12. (10 points) Let A and B be sets such that $|A| = |B| = \aleph_0$. Prove $|A \cup B| = \aleph_0$.
Since $|A| = |B| = \aleph_0$, we can write both A and B in an ordered fashion. So we will consider $A = \{a_1, a_2, a_3, ...\}$ and $B = \{b_1, b_2, b_3, ...\}$. Now we can write $A \cup B$ as $\{a_1, b_1, a_2, b_2, a_3, b_3, ...\}$. This proves that $|A \cup B| = \aleph_0$.

13. (15 points) Use induction to prove one of the following two statements
i. $\sum\limits_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{Z}^+$;
1. Show $S(1)$ is true.
L.H.S.: $\sum\limits_{k=1}^{1} k^3 = 1^3 = 1$
R.H.S. $\frac{1^2(1+1)^2}{4} = 1$ and $S(1)$ is true.
Assume $\sum\limits_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ and show $\sum\limits_{k=1}^{n+1} k^3 = \frac{(n+1)^2(n+2)^2}{4}$.
$\sum\limits_{k=1}^{n+1} k^3 = \sum\limits_{k=1}^{n} k^3 + (n+1)^3$
which by the inductive hypothesis is
$\frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} = \frac{(n+1)^2\left[n^2+4(n+1)\right]}{4} = \frac{(n+1)^2\left[n^2+4n+4\right]}{4} = \frac{(n+1)^2(n+2)^2}{4}$.
or
ii. $\frac{4^{n+1}+5^{2n-1}}{21} \in \mathbb{Z}$ for all $n \in \mathbb{Z}^+$.
$\frac{4^{n+1}+5^{2n-1}}{21} \in \mathbb{Z}$ for all $n \in \mathbb{Z}^+$.
1. Show that $S(1)$ is true.
L.H.S.: $\sum\limits_{k=1}^{1} k2^k = 1 * 2^1 = 2$
R.H.S.: $(1-1)\,2^{1+1} + 2 = 0 + 2 = 2$.
Assume $\sum\limits_{k=1}^{n} k2^k = (n-1)\,2^{n+1} + 2$ and show $\sum\limits_{k=1}^{n+1} k2^k = (n+1-1)\,2^{n+1+1} + 2 = n2^{n+2} + 2$.
$\sum\limits_{k=1}^{n+1} k2^k = \sum\limits_{k=1}^{n} k2^k + (n+1)2^{n+1}$ which by the inductive hypothesis is $(n-1)\,2^{n+1} + 2 + (n+1)2^{n+1} = (n-1+n+1)2^{n+1} + 2 = 2n * 2^{n+1} + 2 = n2^{n+2} + 2$.
2. Show that $S(1)$ is true.
$\frac{4^{1+1}+5^{2*1-1}}{21} = 1 \in \mathbb{Z}$.
Assume $\frac{4^{n+1}+5^{2n-1}}{21} \in \mathbb{Z}$ and show $\frac{4^{n+1+1}+5^{2(n+1)-1}}{21} = \frac{4^{n+2}+5^{2n+1}}{21} \in \mathbb{Z}$.
$\frac{4^{n+2}+5^{2n+1}}{21} = \frac{4*4^{n+1}+25*5^{2n-1}}{21} = \frac{4*4^{n+1}+4*5^{2n-1}+21*5^{2n-1}}{21} = 4 * \frac{4^{n+2}+5^{2n+1}}{21} + \frac{21*5^{2n-1}}{21} = 4 * int + 5^{2n-1} = int$.

14. (10 points) True or False? $\overline{A \oplus B} = \overline{A} \oplus \overline{B}$. If true, prove it. If false, provide a counter example.
False! Let $U = \{1,2,3,4,5,6,7\}$, $A = \{1,2\}$ and $B = \{2,7\}$. Since $A \oplus B = \{1,7\}$, $\overline{A \oplus B} = \{2,3,4,5,6\}$.
On the other hand, $\overline{A} = \{3,4,5,6,7\}$, $\overline{B} = \{1,3,4,5,6\}$ and $\overline{A} \oplus \overline{B} = \{1,7\}$.