Math 3322 Test I
DeMaio Spring 2012

Name____

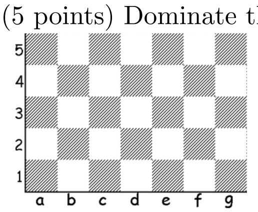
Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (10 points) Compute the following.
   i. $\left\lceil \left(\frac{1}{33}\right)^{250} \right\rceil = 1$
   ii. $\lceil \pi \rceil - \lfloor e \rfloor = 2$
   iii. For f the function that assigns to each bit string, three times the number of 0's in the bit string, evaluate $f(10110101)$.
   $f(10110101) = 3 * 3 = 9$

2. (5 points) True or False? $\lfloor x \lceil y \rceil \rfloor = xy$ for $x, y \in \mathbb{R}$. If true, prove it. If false, provide a counter example.
   False. Let $x = y = \frac{1}{2}$. $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ while $\left\lfloor \frac{1}{2} \left\lceil \frac{1}{2} \right\rceil \right\rfloor = 0$

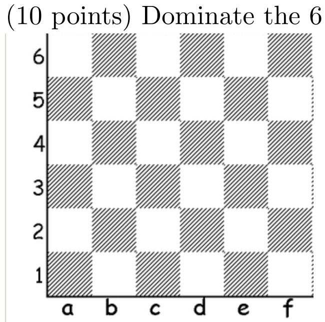
3. (10 points) For even n, place $\frac{n^2}{2}$ non-threatening knights on the $n \times n$ board. Explain why your process always works.
   Place all knights on the black squares. Since a legal move of the knight alternates colors, no knights are threatened and the set is independent.

4. (5 points) Dominate the 5×7 chessboard with 3 queens.

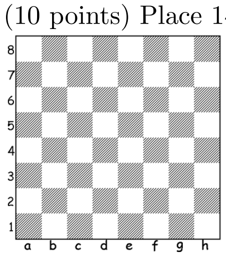
   One possible solution is to place queens on squares $4d$, $3d$ and $2d$.

5. (10 points) Dominate the 6×6 chessboard with 3 queens.

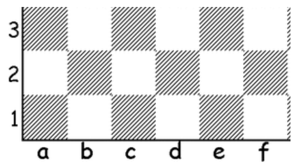
   The unique solution (up to reflections and rotations) is constructed by placing queens on squares $1a$, $5c$ and $3e$.

6. (10 points) Place 14 independent bishops on the 8×8 board.

   One possible solution is to place bishops in the first 7 squares of columns a and h.

7. (10 points) True or False? A closed knights tour of the 3×6 chessboard exists. If true, construct one. If false, explain why it is not possible.



False! Consider square $2a$. A knight has only two legal moves from $2a$: $1c$ and $3c$. The same is true abut square $2e$. Immediately, the closed cycle $2a, 3c, 2e, 1c$ is forced to exist. This cycle does not visit all squares on the board. Hence, no closed knight's tour of the 3×6 board can exist.

8. (10 points) Let f be the function that assigns to each bit string, three times the number of 0's in the bit string. State the domain and range of f.
   The domian is the collection of all bit strings. The range is all non-negative multiples of 3, $\{0, 3, 6, 9, 12, ...\}$.

9. (10 points) Give an example of a function $f : \mathbb{Z}^+ \to \mathbb{Z}^-$ that is neither one-to-one nor onto. Explain why your example is correct. Be certain that your example is a function.
   Consider the constant function $f(x) = -3$. This function is not onto $\mathbb{Z}^-$ since -4 (among other values) has no pre-image. The function os not $1-1$ since $f(4) = f(5) = -3$.

10. (5 points) Why is $f : \mathbb{Z} \to \mathbb{R}$ via $f(x) = \dfrac{1}{\sqrt{x + \frac{1}{2}}}$ not a function? Because the function is undefined at negative values that produce a complex number rather than an element of the codomain.

11. (10 points) In order to admit a closed knight's tour, the number of squares on a rectangular board must be even.
    i. Is this statement *necessary*? Explain.
    Yes! The knight alternates colors with each move. In order to create a closed tour the first square and last square must be different colors. So, you must have the same number of black squares as white squares. This will force the number of squares on the board to be even.
    ii. Is this statement *sufficient*? Explain.
    No! The 3×6 board has an even number of squares but does not admit a closed nights tour.

12. (15 points) i. Let $A = \{a, b, c\}$. Construct $P(A)$.
    $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
    ii. Let B be the set of all bitstrings of length 3. List all the elements of B.
    $\{000, 100, 010, 001, 110, 101, 011, 111\}$
    iii. Create a one-to-one and onto function $f : P(A) \to B$ by providing an explicit rule for the mapping.
    Let $f : P(A) \to B$ by
    placing a 1 in the first position of the bitstring if a is a member of the subset and otherwise placing a 0 if a is not a member of the subset
    and
    placing a 1 in the second position of the bitstring if b is a member of the subset and otherwise placing a 0 if b is not a member of the subset
    and
    placing a 1 in the third position of the bitstring if c is a member of the subset and otherwise placing a 0 if c is not a member of the subset.