Math 3322 Test I Key
DeMaio Summer 2009

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (5 points) Let $A = \{1, 2, 3, 4\}$. True or False: $\emptyset \in A$? Explain. False. While the empty set is a subset of every set it is not necessarily a member of every set. In this case $\emptyset \notin A$.

2. **Homework Section 2.2 #27b** (5 points) Draw a Venn diagram for sets A, B, C and shade the set $(A \cap B) \cup (A \cap C)$.
See homework solutions for picture.

3. (5 points) Compute
i. $\left\lfloor e^{\lceil \pi \rceil} \right\rfloor = 54$.
ii. $\left\lfloor \frac{2}{3} \left\lceil \frac{5}{2} \right\rceil \right\rfloor = 2$.

4. **Homework Section 2.4 #14b** (5 points) Let $S = \{1, 3, 5, 7\}$ and compute $\sum_{j \in S} j^2$. Note $\sum_{j \in S} j^2 = 1^2 + 3^2 + 5^2 + 7^2 = 84$.

5. (5 points) Compute $\sum_{i=56}^{123} i^3$. We want to use the fact that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ but the index of summation in our problem does not begin at $i = 1$. So, $\sum_{i=56}^{123} i^3 = \sum_{i=1}^{123} i^3 - \sum_{i=1}^{55} i^3 = \frac{123^2 * 124^2}{4} - \frac{55^2 * 56^2}{4} = 55\,784\,276$

6. (5 points) Compute $\prod_{k=1}^{10} 2^k$.Well, $\prod_{k=1}^{10} 2^k = 2^1 * 2^2 * ... * 2^{10} = 2^{1+2+...10} = 2^{\frac{10*11}{2}} = 2^{55} = 36\,028\,797$ $018\,963\,968$

7. **Homework Section 2.3 # 4c** (10 points) Find the domain and range of the function that assigns to each bit string the number of ones in that string.
The domain is the collection of all bit strings. The range is $\mathbb{Z}^+ \cup \{0\}$.

8. **Homework Section 2.3 # 12d and 13d** (10 points) Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ via $f(n) = \left\lceil \frac{n}{2} \right\rceil$. Determine, with proof, if $f(n)$ is one-to-one and/or onto.
The function $f(n)$ is not one-to-one since $f(1) = \left\lceil \frac{1}{2} \right\rceil = 1 = \left\lceil \frac{2}{2} \right\rceil = f(2)$. The function $f(n)$ is onto. Let $y \in \mathbb{Z}$. Note that $f(2y) = \left\lceil \frac{2y}{2} \right\rceil = \lceil y \rceil = y$.

9. **Homework Section 2.4 #3d** (10 points) Find the terms a_0, a_1, a_2 and a_3 of the sequence $\{a_n\}$ where $a_n = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil$.
$a_0 = \left\lfloor \frac{0}{2} \right\rfloor + \left\lceil \frac{0}{2} \right\rceil = 0$
$a_1 = \left\lfloor \frac{1}{2} \right\rfloor + \left\lceil \frac{1}{2} \right\rceil = 1$
$a_2 = \left\lfloor \frac{2}{2} \right\rfloor + \left\lceil \frac{2}{2} \right\rceil = 2$
$a_3 = \left\lfloor \frac{3}{2} \right\rfloor + \left\lceil \frac{3}{2} \right\rceil = 3$

10. **Homework Section 2.4 # 10b** (5 points) Find the next three terms of the sequence $3, 6, 11, 18, 27, 38, 51, 66, 83, 102$.
Note that the difference between terms is the sequence of odd numbers starting at 3.

3		6		11		18		27		38		51		66		83		102		?		?		?
	3		5		7		9		11		13		15		17		19		21		23		25	

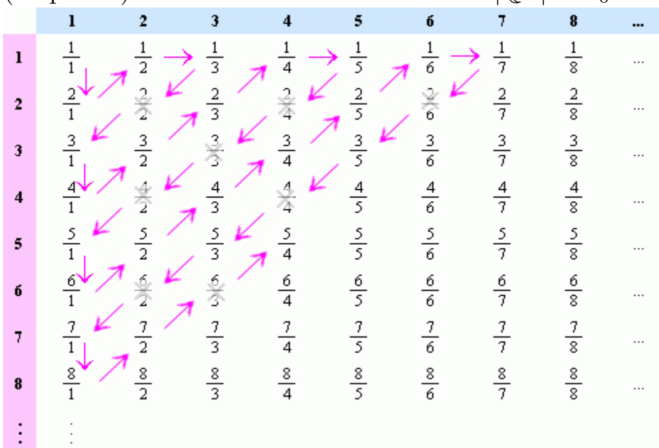
The next terms are $102 + 21 = 123, 123 + 23 = 146$ and $146 + 25 = 171$.

11. (10 points) i.**Homework Section 2.4 #16d** Compute $\sum_{i=0}^{4} \left(2^{i+1} - 2^i\right)$.
With all the dirty details, $\sum_{i=0}^{4} \left(2^{i+1} - 2^i\right) = \left(2^1 - 2^0\right) + \left(2^2 - 2^1\right) + \left(2^3 - 2^2\right) + \left(2^4 - 2^3\right) + \left(2^5 - 2^4\right)$.
Upon closer inspection, we take advantage of the natural cancellation of certain terms and get $\sum_{i=0}^{4} \left(2^{i+1} - 2^i\right) =$

$2^5 - 2^0 = 31$.

ii. Compute $\sum_{i=0}^{30}\left(2^{i+1}-2^i\right)$. Using the logic from part i. we get $\sum_{i=0}^{30}\left(2^{i+1}-2^i\right) = 2^{31} - 2^0 = 2147\,483\,647$.

12. (10 points) **Proof from class** Show that $|\mathbb{Q}^+| = \aleph_0$.



13. (10 points) Use induction to prove one of the following two statements

Homework Section 4.1 #14 and done in class i. $\sum_{k=1}^{n} k2^k = (n-1)\,2^{n+1} + 2$ for all $n \in \mathbb{Z}^+$;

or

ii. $\frac{4^{n+1}+5^{2n-1}}{21} \in \mathbb{Z}$ for all $n \in \mathbb{Z}^+$.

i. Show that $S(1)$ is true.

L.H.S.: $\sum_{k=1}^{1} k2^k = 1 * 2^1 = 2$

R.H.S.: $(1-1)\,2^{1+1} + 2 = 0 + 2 = 2$.

Assume $\sum_{k=1}^{n} k2^k = (n-1)\,2^{n+1} + 2$ and show $\sum_{k=1}^{n+1} k2^k = (n+1-1)\,2^{n+1+1} + 2 = n2^{n+2} + 2$.

$\sum_{k=1}^{n+1} k2^k = \sum_{k=1}^{n} k2^k + (n+1)2^{n+1}$ which by the inductive hypothesis is $(n-1)\,2^{n+1} + 2 + (n+1)2^{n+1} = (n-1+n+1)2^{n+1} + 2 = 2n * 2^{n+1} + 2 = n2^{n+2} + 2$.

ii. Show that $S(1)$ is true.

$\frac{4^{1+1}+5^{2*1-1}}{21} = 1 \in \mathbb{Z}$.

Assume $\frac{4^{n+1}+5^{2n-1}}{21} \in \mathbb{Z}$ and show $\frac{4^{n+1+1}+5^{2(n+1)-1}}{21} = \frac{4^{n+2}+5^{2n+1}}{21} \in \mathbb{Z}$.

$\frac{4^{n+2}+5^{2n+1}}{21} = \frac{4*4^{n+1}+25*5^{2n-1}}{21} = \frac{4*4^{n+1}+4*5^{2n-1}+21*5^{2n-1}}{21} = 4*\frac{4^{n+2}+5^{2n+1}}{21} + \frac{21*5^{2n-1}}{21} = 4*int + 5^{2n-1} = int$.

14. (5 points) **Homework Section 4.3 #24b (close)** Give a recursive definition of the set of positive integer multiples of 3.

Let $t_1 = 3$ and $t_n = 3 + t_{n-1}$ for $t \geq 2$.

15. (5 Bonus points) State the definition of the Fibonacci sequence.

Let $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

16. **Homework on web** (10 Double Bonus points) At a party there are n chairs and some collection of people (including, perhaps none) will sit in the seats but there will always be at least one empty chair between any two people. Let A_n be the number of antisocial ways to seat some number of people in these n seats as described. Construct all possible arrangements and compute A_n for all values up to $n = 4$. Find and prove the correctness of a formula for A_n. Let 0 represent an empty seat and a 1 represent an occupied seat.

$n = 1$	0	$n = 2$	00	$n = 3$	000	$n = 4$	0000
	1		10		001		0001
$A_1 = 2$		$A_2 = 3$	01	$A_3 = 5$	010	$A_4 = 8$	0010
					100		0100
					101		1000
							1001
							1010
							0101

Next we want to show that $A_n = A_{n-1} + A_{n-2}$ for all $n \geq 3$. Partition all A_n seating arrangements into two sets. Set 1 will contain all those arrangements that end in 1 while set 2 will contain all those arrangements that end in 0. There are A_{n-1} seating arrangements in set 2 since we can always place an empty chair at the end of every seating arrangement with $n-1$ chairs. Likewise there are A_{n-2} seating arrangements in set 1 since we can always place an empty chair followed by a filled chair at the end of every seating arrangement with $n-2$ chairs.

17. (10 Double Double Bonus points) Consider the Fibonacci sequence. Using the technique of contradiction, prove that the greatest common divisor of all pairs of consecutive Fibonacci numbers is 1. Begin by assuming that f_k and f_{k+1} are the smallest pair of consecutive Fibonacci numbers such that $\gcd(f_k, f_{k+1}) = d > 1$. Now derive a contradiction.
Since d divides f_k and f_{k+1} we can rewrite $f_k = dr$ and $f_{k+1} = ds$ for some $r, s \in \mathbb{Z}^+$. By definition of the Fibonacci sequence $f_{k+1} = f_k + f_{k-1}$ which implies $dr = ds + f_{k-1}$. Rewrite this as $f_{k-1} = dr - ds = d(r - s)$ we see that d divides f_{k-1}. This is a problem because now $\gcd(f_k, f_{k-1}) \geq d > 1$ which contradicts the fact that f_k and f_{k+1} are the smallest pair of consecutive Fibonacci numbers such that $\gcd(f_k, f_{k+1}) = d > 1$. Thus no consecutive Fibonacci numbers exist whose gcd is larger than 1.