Math 3322 Test I
DeMaio Spring 2009

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. Where appropriate, simplify all answers to a single decimal expansion.

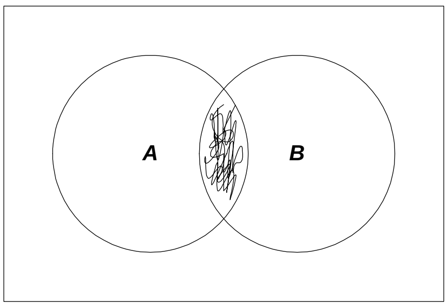
Preliminaries

1. (5 points) List the members of the set $S = \{x | x \text{ is the square of an integer and } x < 100\}$.
   $S = \{0, 1, 4, 9, 16, 25, 36, 49, 64, 81\}$

2. (5 points) Compute $|A|$ for $A = \mathbb{Z}^+ \cup \mathbb{Q}^-$.
   $|A| = \aleph_0$

3. (5 points) i. Give an example of infinite sets A and B such that $|A - B| = \aleph_0$. Let $A = \mathbb{Z}$ and $B = \mathbb{Z}^+$.
   (5 points) ii. Give an example of infinite sets A and B such that $|A - B| = 0$. Let $A = \mathbb{Z}$ and $B = \mathbb{R}$.

4. (5 points) In a Venn diagram, shade $\overline{A \oplus B} \cap A$.



5. (5 points) Let A be the set of students who live within one mile of campus. Let B be the set of all students who walk to class. What does it mean to say Mary is a member of the set $\overline{A} \cap \overline{B}$? Mary lives more than one mile from campus and she does not walk to class.

6. (5 points) Compute $\left\lfloor 5.68 + \left\lfloor \frac{1}{2} + \left\lceil \frac{1}{2} \right\rceil \right\rfloor + \frac{3}{4} \right\rfloor = 7$.

7. (5 points).Let $A = \{x | x \text{ is the square of an integer and } x < 100\}$ and
   $B = \{x | x \text{ is the cube of an integer and } x < 100\}$. Find $A \cap B$.
   $A \cap B = \{0, 1, 64\}$

8. (5 points) Compute $\prod_{k=1}^{100} (-1)^k = (-1)^{50} = 1$.

9. (5 points) Compute $\sum_{i \in S} i^2$ for $S = \{-3, 1, 0, 2\}$.
   $\sum_{i \in S} i^2 = (-3)^2 + 1^2 + 0^2 + 2^2 = 14$

10. (5 points) Compute $\sum_{i=1}^{278} \left(1 + (-1)^i\right) = 278$.

11. (5 points) Compute $\frac{100!}{95!5!} = 75\,287\,520$.

12. (5 points) Find the first 10 terms of the sequence whose first two terms are 1 and 2 and each succeeding term is the sum of the previous two terms.
    $1, 2, 3, 5, 8, 13, 21, 34, 55, 89$

13. (5 points) Is $\lfloor j + k \rfloor = \lfloor j \rfloor + \lceil k \rceil$? If yes, prove it. If no, provide a counter example.
    False! Let $j = k = 1.2$. Now $\lfloor j + k \rfloor = \lfloor 2.4 \rfloor = 2$ while $\lfloor j \rfloor + \lceil k \rceil = \lfloor 1.2 \rfloor + \lceil 1.2 \rceil = 3$.

14. (10 points) Let $f : \mathbb{Z} \to \mathbb{Z}^+$ via $f(n) = \left\lceil \frac{n^2}{2} \right\rceil$.

a. Is $f(n)$ onto? Explain.

No. The integer 3 has no pre-image since $\left\lceil \frac{1^2}{2} \right\rceil = 1$, $\left\lceil \frac{2^2}{2} \right\rceil = 2$, $\left\lceil \frac{3^2}{2} \right\rceil = 5$ and $f(n)$ is an increasing function on $\mathbb{Z}^+$ and decreasing on $\mathbb{Z}^-$.

b. Is $f(n)$ one to one? Explain.
No since $f(-2) = f(2)$.

15. (10 points) Do one of the following two induction problems. Clearly indicate your selection.

a. Find and prove the correctness of a formula for $\sum_{i=1}^{n} 2i$.

We need to conjecture and closed form function for $\sum_{i=1}^{n} 2i$. So, let's plug and chug a few values until we see a pattern.

n	$\sum_{i=1}^{n} 2i$	???
1	$\sum_{i=1}^{1} 2i = 2$	$1 * 2$
2	$\sum_{i=1}^{2} 2i = 6$	$2 * 3$
3	$\sum_{i=1}^{3} 2i = 12$	$3 * 4$
4	$\sum_{i=1}^{4} 2i = 20$	$4 * 5$

So, it appears that $\sum_{i=1}^{n} 2i = n(n+1)$. The above table serves as our base cases. So, assume that $\sum_{i=1}^{n} 2i = n(n+1)$ and show $\sum_{i=1}^{n+1} 2i = (n+1)(n+2)$.

$\sum_{i=1}^{n+1} 2i = \sum_{i=1}^{n} 2i + 2(n+1) = n(n+1) + 2(n+1) = (n+1)(n+2)$. Put both parts together and $\sum_{i=1}^{n} 2i = n(n+1)$ for all $n \in \mathbb{Z}^+$.

b. Prove $\frac{5^{n+2}+9^n+10}{4}$ is an integer for all $n \in \mathbb{Z}^+$.
First note that $\frac{5^{n+2}+9^n+10}{4} = \frac{5^{1+2}+9^1+10}{4} = 36 \in \mathbb{Z}$ and $S(1)$ is true.
Next assume $\frac{5^{n+2}+9^n+10}{4}$ is an integer and show $\frac{5^{n+1+2}+9^{n+1}+10}{4}$ is an integer.
$\frac{5^{n+1+2}+9^{n+1}+10}{4} = \frac{5^{n+3}+9^{n+1}+10}{4} = \frac{5*5^{n+2}+9*9^n+10}{4} = \frac{5^{n+2}+9^n+10}{4} + \frac{4*5^{n+2}}{4} + \frac{8*9^n}{4} = int + 5^{n+2} + 2*9^n = int$. Put both parts together and $\frac{5^{n+2}+9^n+10}{4}$ is an integer for all $n \in \mathbb{Z}^+$.

16. (10 points) Use the strong form of induction to prove that any amount of postage $n \geq 30$ cents can be made using only 4 cent and 11 cent stamps.
Four base cases are required since the smallest denomination is 4.
$30¢ = 2 * 11¢ + 2 * 4¢$
$31¢ = 11¢ + 5 * 4¢$
$32¢ = 8 * 4¢$
$33¢ = 3 * 11¢$
So, the statement is true for $S(30), S(31), S(32)$ and $S(33)$. Now assume $S(30), S(31), ..., S(n-1)$ and $S(n)$ are all true for $n \geq 33$. Show $S(n+1)$ is also true.
Well, $(n+1)¢ = 4¢ + (n-3)¢$. By the inductive assumption we know that $(n-3)¢$ of postage can be made using only 4 cent and 11 cent stamps since $n \geq 33$ which forces $n-3 \geq 30$ which is covered

by the base cases or subsequent inductive iterations. By parts 1 and 2 any amount of postage $n \geq 30$ cents can be made using only 4 cent and 11 cent stamps.

17. (5 points) Find the error in the following proof of this "theorem":

    "Theorem: Every positive integer equals the next largest positive integer."

    " Proof: Let $P(n)$ be the proposition '$n = n+1$'. To show that $P(k) = P(k+1)$, assume that $P(k)$ is true for some k, so that $k = k+1$. Add 1 to both sides of this equation to obtain $k+1 = k+2$, which is $P(k+1)$. Therefore $P(k) = P(k+1)$ is true. Hence $P(n)$ is true for all positive integers n.".
    No base case exists!!!

18. (10 points) For sets A, B and C, show $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$.
    First show $\overline{A \cap B \cap C} \subseteq \overline{A} \cup \overline{B} \cup \overline{C}$. Let $x \in \overline{A \cap B \cap C}$. Thus, $x \notin A \cap B \cap C$ and x is not an element of at least one of A, B or C. Thus, x is an element of at least one of $\overline{A}, \overline{B}$ or $\overline{C}$.This shows $x \in \overline{A} \cup \overline{B} \cup \overline{C}$.
    Next show $\overline{A} \cup \overline{B} \cup \overline{C} \subseteq \overline{A \cap B \cap C}$. Let $x \in \overline{A} \cup \overline{B} \cup \overline{C}$. So, x is not a member of at least one of A, B or C. Without loss of generality we can say $x \notin A$. Hence, $x \notin A \cap B \cap C$ which shows that $x \in \overline{A \cap B \cap C}$.
    Put the two parts together and $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$.