Math 3322 Test II
DeMaio Spring 2010

Name______________________________

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (30 points) Recall our 20 person club with 13 women and 7 men. How many ways can we
   i. elect a President and Vice-President; $20 * 19 = 380$
   ii. form a committee of three people (where each member has equal rank and power); $\binom{20}{3} = 1140$
   iii. form a committee of four people and elect one of those four members to be the chairperson; $\binom{20}{4} * 4 = 19\,380$
   iv. form a committee of two people of opposite gender; $13 * 7 = 91$
   v. form a committee of two people of the same gender; $\binom{13}{2} + \binom{7}{2} = 99$ or $\binom{20}{2} - 91 = 99$
   vi. form a committee of six people with at least one member of each gender? $\binom{20}{6} - \binom{13}{6} - \binom{7}{6} = 37\,037$

2. (15 points) How many ways can we rearrange the letters in the word
   i. vampire; $7! = 5040$
   ii. werewolf; $\frac{8!}{2! * 2!} = 10\,080$
   iii. mummy? $\frac{5!}{3!} = 20$

3. (10 points) How many positive integers not exceeding 10,000 are divisible by 6 or 15? $\left\lfloor \frac{10000}{6} \right\rfloor + \left\lfloor \frac{10000}{15} \right\rfloor - \left\lfloor \frac{10000}{30} \right\rfloor = 1999$

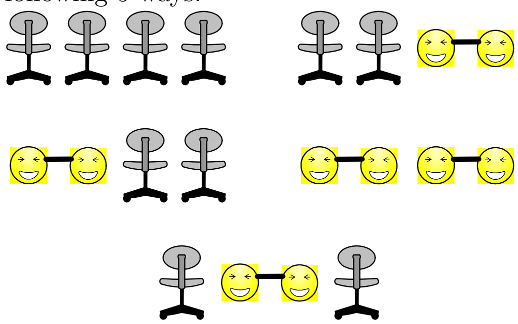
4. (10 points) Use the Binomial Theorem to expand $(3x-2)^5$. You must show all the details of your work.
   $(3x-2)^5 =$
   $\binom{5}{5}(3x)^5(-2)^0 + \binom{5}{4}(3x)^4(-2)^1 + \binom{5}{3}(3x)^3(-2)^2 + \binom{5}{2}(3x)^2(-2)^3 + \binom{5}{1}(3x)^1(-2)^4 + \binom{5}{0}(3x)^0(-2)^5$
   $= 243x^5 - 5 * 81 * 2x^4 + 10 * 27 * 4x^3 - 10 * 9 * 8x^2 + 5 * 3 * 16x - 32$
   $= 243x^5 - 810x^4 + 1080x^3 - 720x^2 + 240x - 32$

5. (10 points) John has 13 ordinary coins (cent, nickel, dime, quarter) in his pocket.
   At least how many of the same coin must John have? $\left\lceil \frac{13}{4} \right\rceil = 4$
   At least how many quarters must John have? None.

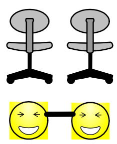
6. (10 points) What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state? $50 * 99 + 1 = 4951$.

7. (15 points) Use induction to prove $3^n < n!$ for integers $n \geq 7$.
   First show $S(7)$ is true: $3^7 = 2187 < 5040 = 7!$
   Second assume $3^n < n!$ and show $3^{n+1} < (n+1)!$.
   So, $3^{n+1} = 3 * 3^n < 3 * n!$ by the inductive assumption. Since $n \geq 7$, $n > 3$ and $3 * n! < (n+1) * n! = (n+1)!$. Thus, $3^{n+1} < (n+1)!$.

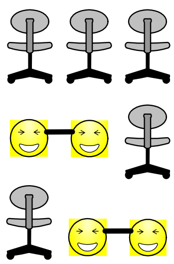
8. (15 points) Chairs in a Row: the Couples version: In this version, we have a row of n chairs. People arrive in twos and want to sit next to their partner. For instance, a row of 4 chairs can be filled in the following 5 ways:



Find all the ways a row of 2 chairs can be filled with couples or be left empty?



Find all the ways a row of 3 chairs can be filled with couples or be left empty?



Conjecture and prove a formula for the number of ways a row of n chairs can be filled with couples or be left empty? Let S_n be the number of ways a row of n chairs can be filled with couples or be left empty. Given that $S_2 = 2$, $S_3 = 3$ and $S_4 = 5$ it would appear that $S_n = F_{n+1}$. So, is $S_n = S_{n-1} + S_{n-2}$? Partition all S_n rows into two sets, those that end in an empty chair and those that end in a person. There are S_{n-1} that end in an empty chair because you can add an empty chair on the end of every row of length $n-1$. A row that ends in a person, actually ends in two people since couples sit together. This means there are S_{n-2} rows that end in a person since you can add a couple to the end of every row of length $n-2$. Thus, $S_n = S_{n-1} + S_{n-2}$ and $S_n = F_{n+1}$.