Math 3322 Test II
DeMaio Fall 2012

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

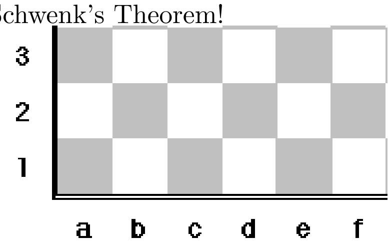
1. (30 points) In the questions below suppose that a "word" is any string of five uppercase letters of the alphabet, with repeated letters allowed unless otherwise noted.. Also, letters are available in both blue and red colors. There are an infinite number of each color letter.
   i. How many words exist? $52^5 = 380\,204\,032$
   ii. How many words are all red and end with the letter B? $26^4 = 456\,976$
   iii. How many words begin with a red A and end with a blue B? $52^3 = 140\,608$
   iv. How many words begin with A or a blue letter? $27 * 52^4 = 197\,413\,632$
   v. How many words begin with A or end with B? $2 * 52^4 + 2 * 52^4 - 2 * 52^3 * 2 = 28\,684\,032$
   vi. How many words have no consecutive letters that are the same color? $2 * 26^5 = 23\,762\,752$

2. (10 points) Write a recursive definition of the positive odd integers.
   Let $O_n = O_{n-1} + 2$ for $n \geq 2$ where $O_1 = 1$.

3. (10 points) Let $f(0) = 3$ and $f(n) = 2^{f(n-1)-2}$ for $n \geq 1$. Compute the following.
   i. $f(1) = 2^{3-2} = 2$
   ii. $f(2) = 2^{2-2} = 1$
   iii. $f(3) = 2^{1-2} = \frac{1}{2}$
   iv. $f(4) = 2^{\frac{1}{2}-2} = \frac{1}{4}\sqrt{2}$

4. (15 points) Prove that any amount of postage ≥ 18 cents can be made using only 4 cent and/or 7 cent stamps.
   $18 = 2 * 7 + 4$
   $19 = 7 + 3 * 4$
   $20 = 5 * 4$
   $21 = 3 * 7$
   Assume that any amount of postage $18, 19, 20, 21, ..., n$ cents for $n \geq 21$ can be made using only 4 cent and/or 7 cent stamps. Show how to make $n + 1$ cents of postage.
   $n + 1 = 4 + n - 3$
   Note that $n - 3 \leq n$ and since $n \geq 21$ then $n - 3 \geq 18$.
   Thus, by inductive assumption we can make $n - 3$ cents worth of postage using only 4 cent and/or 7 cent stamps. Add a 4 cent stamp to that solution and we've made $n + 1$ cents of postage using only 4 cent and/or 7 cent stamps.

5. (15 points) Prove that the 3×6 board does not admit a closed knight's tour. You may not reference Schwenk's Theorem!



6. (15 points) Use the fact that the $\gcd(F_n, F_{n+1}) = 1$ to prove $\gcd(F_n, F_{n+2}) = 1$.
   Assume $\gcd(F_n, F_{n+2}) = d > 1$ for some n. Since $F_{n+2} = F_n + F_{n+1}$ then $F_{n+1} = F_{n+2} - F_n$. Since $\gcd(F_n, F_{n+2}) = d$ then d divides both F_n and F_{n+2}. Thus, d divides their difference, F_{n+1}. However, now d divides both F_n and F_{n+1} so $1 = \gcd(F_n, F_{n+2}) \geq d > 1$ which is a problem. Hence, the assumption is wrong and $\gcd(F_n, F_{n+2}) = 1$.

7. (15 points) Determine a formula for $\sum_{i=1}^{n}(F_i)^2$. Use induction to prove the correctness of your conjecture.

   Prove $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$ for the Fibonacci sequence and $n \in Z^+$.
   First show that the base case of $n = 1$ hold true.
   R.H.S. $\sum_{i=1}^{1} F_i^2 = F_1^2 = 1^2 = 1$ while L.H.S. $F_1 F_{1+1} = F_1 F_2 = 1 * 1 = 1$
   Assume the statement is true for $n : \sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$. Show that the statement is true for the value $n+1 : \sum_{i=1}^{n+1} F_i^2 = F_{n+1} F_{n+2}$.
   Note that $\sum_{i=1}^{n+1} F_i^2 = \sum_{i=1}^{n} F_i^2 + F_{n+1}^2$ which by the inductive hypothesis is $F_n F_{n+1} + F_{n+1}^2 = F_{n+1}(F_n + F_{n+1}) = F_{n+1} F_{n+2}$.