Math 4322 Test II
DeMaio Spring 2010

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (10 points) State Kuratowski's Theorem.
A graph G is planar if and only if it does not contain a subgraph homeomorphic to $K_{3,3}$ or $K_{5.}$
Determine, with proof, if $K_{2,2,2}$ is planar.

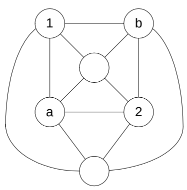

Figure 1:
$K_{2,2,2}$ is Planar

2. (10 points) Compute the following.
i. $\chi(K_n) = n$
ii. $\chi(C_n) = \left\{ \begin{array}{c} 2 \text{ if } n \text{ is even} \\ 3 \text{ if } n \text{ is odd} \end{array} \right\}$
iii. $\chi(K_{n_1,n_2,\ldots,n_r}) = r$

3. (5 points) Determine $\chi\left(\overline{K_{m,n}}\right) = \max\{m, n\}$

4. (10 points) Andy is going to set up some aquarium tanks at home for six species of tropical fish. Naturally, Andy does not wish to house any species that would prey on each other in the same tank. Suppose species 1 feeds on species 2 and 5; species 2 feeds on species 1 and 5; species 3 feeds on species 4 and 5; species 4 feeds on species 2; and species 5 and 6 do not feed on any other species. Use graph theory to determine, with proof, the minimum number of tanks Andy will need.
We first construct the graph representing the 6 species. Each species is a vertex. Two vertices (species) are adjacent if they cannot be in the same tank. The resulting graph is shown in Figure 2. The triangle formed by vertices 1, 2 and 5 show that $\chi(G) \geq 3$. The 3-coloring of Figure 3 shows $\chi(G) \leq 3$. Thus, $\chi(G) = 3$ and Andy needs a minimum of 3 tanks.

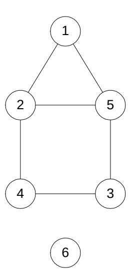

Figure 2

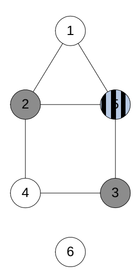

Figure 3:
A
3-coloring
of Figure
2

5. (10 points) Prove that Q_n is Hamiltonian for all $n \geq 2$.
We proceed by inducting on n. Since $Q_2 \cong C_4$ and every cycle graph is Hamiltonian it is clear that $S(2)$ is true. Now assume that Q_n is Hamiltonian and show that Q_{n+1} is Hamiltonian. Take two copies of the Hamiltonian cycle on Q_n. On the first copy append a 0 to every vertex label while on the second copy append a 1 to every label. We now have all the vertices of Q_{n+1}. Take any $a-b$ edge in the first copy of Q_n. In Q_{n+1} we have two occurrences of this edge:once as $a0-b0$ and again as $a1-b1$. Delete the $a0-b0$ and $a1-b1$ edges. Note that $a0$ and $a1$ differ only in the last bit and are adjacent in Q_{n+1}. Ditto for $b0$ and $b1$. We can create the $a0-a1$ and $b0-b1$ edges. We now have a Hamiltonian cycle in Q_{n+1}.

6. (10 points) Let $P(G,x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_2x^2 + a_1x$ be the chromatic polynomial for a graph with a non-empty edge set. Prove $\sum_{i=1}^{n} a_i = 0$.
On the one hand, $P(G,1) = a_n1^n + a_{n-1}1^{n-1} + a_{n-2}1^{n-2} + ... + a_21^2 + a_11 = \sum_{i=1}^{n} a_i$. On the other hand if G contains an edge then at least two colors are needed to color G. This forces $P(G,1) = 0$. We've evaluated $P(G,1)$ in two different ways and thus, $\sum_{i=1}^{n} a_i = 0$.

7. (15 points) For each of the following polynomials, determine if it is a chromatic polynomial for some graph. If it is, draw such a graph. If not, explain why not.
i. The polynomial $x^3 - 3x^2 + 4x - 2$ is not a chromatic polynomial because the constant term is not 0.
ii. The polynomial $x^6 - x^5 = x^5(x-1)$ is the chromatic polynomial of a graph that consists of P_2 and four additional isolated vertices.
iii. The polynomial $x^5 - 12x^4 + 6x^3 - 2x^2 + 7x$ is not a chromatic polynomial because the coefficient of x^4 is -12 which indicates that the graph has 12 edges. However, the degree of the polynomial is 5 which indicates there are 5 vertices. This is a problem since a graph with 5 vertices has a maximum of $\binom{5}{2} = 10$ edges.

8. (15 points) State Dirac's Theorem. Let $G = (V, E)$ be a graph with n vertices. If $\deg(v) \geq \frac{n}{2}$ for every $v \in V$ then G is Hamiltonian.
Determine, with proof, when $K_{n_1,n_2,...,n_r}$ is Hamiltonian.
Note that $K_{n_1,n_2,...,n_r}$ has $n = \sum_{i=1}^{r} n_i$ vertices. Without loss of generality assume $n_i \leq n_{i+1}$ for all $i = 1, 2, ..., r$. The minimum degree of a vertex in G is the degree of the vertex in the n_r^{th}partite set which is $\sum_{i=1}^{r-1} n_i$. When is $\sum_{i=1}^{r-1} n_i \geq \frac{n}{2}$? Since $n = \sum_{i=1}^{r} n_i$,
$\sum_{i=1}^{r-1} n_i \geq \frac{\sum_{i=1}^{r} n_i}{2}$
$n - n_r \geq \frac{n}{2}$
$\frac{n}{2} \geq n_r$.
So, as long as $\frac{n}{2} \geq n_r$, G is Hamiltonian. Of course, Dirac's Theorem is not an if and only if statement. Can G be Hamiltonian if $\frac{n}{2} > n_r$? No! There will not be enough vertices in the first $r-1$ sets to allow leaving and entering the r^{th} partite set when visiting every vertex exactly once.

9. (10 points) Determine the chromatic polynomial of the following graph G.

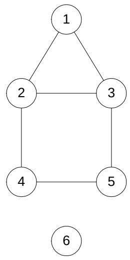

Figure 4

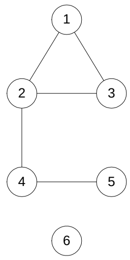

Figure 5: G_1

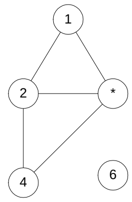

Figure 6: G_2

Figure 4 contains a C_4 and we will need to use the chromatic polynomial reduction formula. We will use the $3-5$ edge to construct G_1 and G_2. The chromatic polynomials of G_1 and G_2 can be computed using the multiplication rule. Note that $P(G_1, x) = x^2 (x-1)^3 (x-2)$ and $P(G_2, x) = x^2 (x-1)(x-2)^2$. Thus, $P(G, x) = x^2 (x-1)^3 (x-2) - x^2 (x-1)(x-2)^2 = x^6 - 6x^5 + 14x^4 - 15x^3 + 6x^2$.

10. (10 points) Prove that the 3×6 chessboard does not admit a closed knight's tour.

1	4	7	10	13	16
2	5	8	11	14	17
3	6	9	12	15	18

Vertices 2 and 14 have degree 2 and immediately force the closed cycle $2-7-14-9-2$ with no opportunity to include the other 14 vertices. Thus, no closed knight's tour exists.

11. (5 points) With a brief proof, show no Eulerian circuit exists on the $4 \times n$ chessboard for $n \geq 3$.

1	5	9	...	4n-3
2	6	10	...	4n-2
3	7	11	...	4n-1
4	8	12		4n

In an Eulerian graph, the degree of every vertex is even. Note however, that square 2 has degree 3.