Math 3322 Test II
DeMaio Fall 2011

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (35 points) In the questions below suppose that a “word” is any string of five uppercase letters of the alphabet, with repeated letters allowed unless otherwise noted. Furthermore, always classify the pesky "sometimes Y" as a consonant.
   i. How many words exist? $26^5 = 11\,881\,376$
   ii. How many words end with the letter B? $26^4 = 456\,976$
   iii. How many words begin with A and end with B? $26^3 = 17\,576$
   iv. How many words begin with A or B and have no repeated letters? $2 * 25 * 24 * 23 * 22 = 607\,200$
   v. How many words begin with A or end with B? $26^4 + 26^4 - 26^3 = 896\,376$
   vi. How many words have no vowels? $21^5 = 4084\,101$
   vii. How many words have exactly two vowels? $\binom{5}{2} * 5^2 * 21^3 = 2315\,250$

2. (5 points) How many ways can 6 people be seated around a circular table if left and right sides are considered different? $(6-1)! = 120$

3. (5 points) How many ways can 10 people be seated around a circular table if left and right sides are not considered different and Alice and Bill must sit beside each other? We group Alice and Bill together as one block. This can be done in two ways, AB or BA. Next arrange 9 objects around the table. Divide by 2 to remove the repetition since left and right are irrelevant here. So, $\frac{2*(9-1)!}{2} = 40\,320$ ways.

4. (5 points) How many functions exist from a domain with 7 elements to a codomain with 10 elements? $10^7 = 10\,000\,000$

5. (5 points) How many one-to-one functions exist from a domain with 7 elements to a codomain with 10 elements? $\frac{10!}{3!} = 604\,800$

6. (10 points) i. How many bits strings are there of length six or less, not counting the empty string? $2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 126$
   ii. How many bits strings are there of length ten with exactly six zeroes? $\binom{10}{6} = 210$

7. (10 points) Use sophisticated counting techniques to determine how many positive integers not exceeding 1,000 are divisible by neither 9 nor 12?
   $\left\lfloor \frac{1000}{9} \right\rfloor + \left\lfloor \frac{1000}{12} \right\rfloor - \left\lfloor \frac{1000}{\mathrm{lcm}(9,12)} \right\rfloor = 167$ are divisible by 9 or 12. So, $1000 - 167 = 833$ are divisible by neither 9 nor 12.

8. (10 points) A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.
   i. How many socks must he take out to be sure that he has at least two socks of the same color? Explain. Three. With 3 pigeons (socks) and 2 pigeonholes (sock color) we know that at least one sock color appears at least $\left\lceil \frac{3}{2} \right\rceil = 2$ times for a pair of matched socks.
   ii. How many socks must he take out to be sure that he has at least two black socks? Explain. 14. He might take out all 12 brown socks first and then get the two desired black socks since there are no other choices.

9. (5 points) What is the minimum number of students, each of whom comes from one of the fifty states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state? $99 * 50 + 1 = 4951$

10. (10 points) Use algebra to prove $\binom{n}{k} = \binom{n}{n-k}$ for all integers $0 \le k \le n$.
    $\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!n!} = \binom{n}{k}$.

11. (15 points) Determine a formula for $\sum_{i=1}^{n} (F_i)^2$. Use induction to prove the correctness of your conjecture.

Checking a few values of n, we get:

n	sum	factor
1	$1 = 1$	$1 * 1$
2	$1 + 1 = 2$	$1 * 2$
3	$1 + 1 + 4 = 6$	$2 * 3$
4	$1 + 1 + 4 + 9 = 15$	$3 * 5$
5	$1 + 1 + 4 + 9 + 25 = 40$	$5 * 8$

It appears that $\sum_{i=1}^{n} (F_i)^2 = F_n F_{n+1}$. Now we proceed with induction to prove the correctness of our conjecture. We have already shown that the statement is true for $n = 1, 2, 3, 4, 5$ while determining the formula for our conjecture. Assume $\sum_{i=1}^{n} (F_i)^2 = F_n F_{n+1}$ and show $\sum_{i=1}^{n+1} (F_i)^2 = F_{n+1} F_{n+2}$.

So, $\sum_{i=1}^{n+1} F_i^2 = \sum_{i=1}^{n} F_i^2 + F_{n+1}^2$ which by the inductive hypothesis is $F_n F_{n+1} + F_{n+1}^2 = F_{n+1} (F_n + F_{n+1})$. By the definition of the Fibonacci sequence $F_{n+1} (F_n + F_{n+1}) = F_{n+1} F_{n+2}$ as desired.