Math 3322 Test II
DeMaio Spring 2012

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. Where appropriate, simplify all answers to a single decimal expansion.

1. (15 points) Let $g_0 = 1$. Let $g_n = 1 + 2^{g_{n-1}}$ for $n \geq 1$. Compute g_1, g_2, g_3 and g_4.
   $g_1 = 1 + 2^{g_0} = 1 + 2^1 = 3$
   $g_2 = 1 + 2^{g_1} = 1 + 2^3 = 9$
   $g_3 = 1 + 2^{g_2} = 1 + 2^9 = 513$
   $g_4 = 1 + 2^{g_3} = 1 + 2^{513}$

2. (10 points) Give a recursive definition of the set of positive integer powers of 5.
   Let $P_1 = 5$. Let $P_n = 5P_{n-1}$ for $n \geq 2$.

3. (10 points) State the recursive definition of the Fibonacci sequence.
   Let $f_0 = 0$ and $f_1 = 1$. For $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$.

4. (10 points) Complete the table of Fibonacci numbers.

n	0	1	2	3	4	5	6	7	8	9	10
f_n	0	1	1	2	3	5	8	13	21	34	55

5. (15 points) Use induction to prove $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$ for the Fibonacci sequence and $n \in Z^+$.
   First show that the base case of $n = 1$ hold true.
   R.H.S. $\sum_{i=1}^{1} F_i^2 = F_1^2 = 1^2 = 1$ while L.H.S. $F_1 F_{1+1} = F_1 F_2 = 1 * 1 = 1$
   Assume the statement is true for $n : \sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$. Show that the statement is true for the value $n + 1 : \sum_{i=1}^{n+1} F_i^2 = F_{n+1} F_{n+2}$.
   Note that $\sum_{i=1}^{n+1} F_i^2 = \sum_{i=1}^{n} F_i^2 + F_{n+1}^2$ which by the inductive hypothesis is $F_n F_{n+1} + F_{n+1}^2 = F_{n+1}(F_n + F_{n+1}) = F_{n+1} F_{n+2}$.

6. (15 points) Use induction to prove $3^n < n!$ for integers $n \geq 7$.
   Since $3^7 = 2187$ and $7! = 5040$ the statement is true for $n = 7$.
   Assume $3^n < n!$ and show $3^{n+1} < (n+1)!$.
   Note that $3^{n+1} = 3 * 3^n$. By the inductive hypothesis $3 * 3^n < 3 * n!$. Since $3 < 7 \leq n < n + 1$, we know that $3 * n! < (n + 1) * n! = (n + 1)!$. String these statements together and $3^{n+1} < (n + 1)!$.

7. (15 points) Use induction to show $\frac{2^{3n}-22}{7}$ is an integer for $n \in \mathbb{Z}^+$.
   First, show $S(1)$ is true. Note, $\frac{2^{3*1}-22}{7} = -2$ which is an integer.
   Assume $\frac{2^{3n}-22}{7}$ is an integer and show $\frac{2^{3(n+1)}-22}{7} = \frac{2^{3n+3}-22}{7}$ is an integer.
   Well, $\frac{2^{3n+3}-22}{7} = \frac{8*2^{3n}-22}{7} = \frac{2^{3n}-22}{7} + \frac{7*2^{3n}}{7} = \frac{2^{3n}-22}{7} + 2^{3n}$. The first part of the sum is an integer by the inductive hypothesis and 2^{3n} is clearly an integer for all integers n. The sum of two integers is an integer. Thus, $\frac{2^{3n+3}-22}{7}$ is an integer.

8. (20 points) Use the closed knight's tour of the 3×12 board of Figure 1 and the open 3×4 board of Figure 2 to prove the existence of a closed knight's tour for all $3 \times n$ boards where 4 divides n. Feel free to add whatever labels you like to the squares of the board.

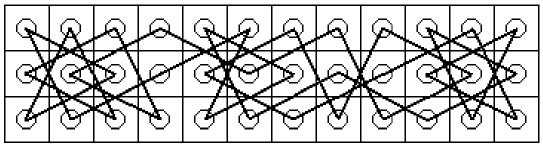

Figure 1

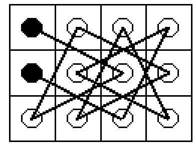

Figure 2