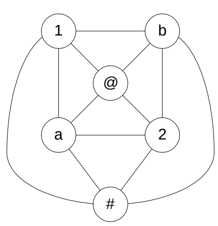
Math 4322 Test 2
DeMaio Spring 2009

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. Where appropriate, simplify all answers to a single decimal expansion.

1. (40 points) Compute the following.
   i. $\omega(K_n) = n$
   ii. $\omega(C_n) = 2$ for $n \geq 4$ but 3 when $n = 3$
   iii. $\beta_0(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$
   iv. $\beta_0(K_{n_1,n_2,\ldots,n_r}) = \max\{n_i\}$
   v. $\gamma(K_{n_1,n_2,\ldots,n_r}) = 2$
   vi. $\gamma_t(K_{n_1,n_2,\ldots,n_r}) = 2$
   vii. $\chi(K_{n_1,n_2,\ldots,n_r}) = r$
   viii. $\beta_0\left(\overline{K_{n_1,n_2,\ldots,n_r}}\right) = r$
   ix. The number of unlabeled trees with 7 vertices is $7^{7-2} = 16\,807$
   x. The number of edges in a tree with 10,000 vertices is 9999.

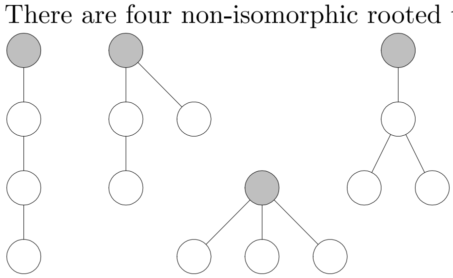
2. (10 points) Determine, with proof, if $K_{2,2,2}$ is planar.
   The graph $K_{2,2,2}$ is planar as shown below.



3. (10 points) Let $T = (V, E)$ be a tree. Prove for all $a, b \in V$ there exists a unique $a - b$ path in T.
   Let $a, b \in V$. We know that an $a - b$ path exists since all trees are connected. Is this path unique? Consider if two different $a - b$ paths in T exist. If two such paths exist they can be joined together to form a circuit in T. This contradicts the fact that T contains no cycles. So, two different $a - b$ paths in T do not exist. Thus, the $a - b-$ path is unique.

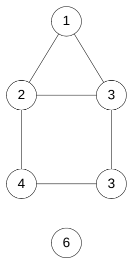
4. (15 points) Let $T = (V, E)$ be a tree with $e \neq 0$ edges. Find $\chi(T)$ and $P(T, x)$. With four colors available how many different legal colorings of a tree with 7 edges exist?
   With a non-zero number of edges $\chi(T) = 2$ and $P(T, x) = x(x-1)^e = P(T, x) = x(x-1)^{n-1}$. With four colors available there are $P(T, 4) = 4(4-1)^7 = 8748$ different legal colorings of a tree with 7 edges.

5. (10 points) Find all non-isomorphic rooted trees with 4 vertices.
   There are four non-isomorphic rooted trees with 4 vertices.



6. (10 points) Let S be a maximal independent set of a graph G. Prove that S is also a dominating set of G. If a vertex v is not dominated by S then $S \cup \{v\}$ is also independent. However, S is a maximal independent set. Thus, no such v may exist and S is also a dominating set.

7. (10 points) Andy is going to set up some aquarium tanks at home for six species of tropical fish. Naturally, Andy does not wish to house any species that would prey on each other in the same tank. Suppose species 1 feeds on species 2 and 5; species 2 feeds on species 1 and 5; species 3 feeds on species 4 and 5; species 4 feeds on species 2; and species 5 and 6 do not feed on any other species. Use graph theory to determine, with proof, the minimum number of tanks Andy will need.
Color vertices 1 and 4 blue, 2 and 3 red and 5 and 6 green. Such a legal coloring shows that Andy needs at most 3 tanks. The triangle form by vertices 1, 2 an5 show that Andy needs at least 3 tanks. Thus, the minimum number of tanks needed is 3.



8. (15 points) Let $G = (V, E)$ be a graph. As described in the Supplementary Exercises define what it means for a property to be monotone increasing and monotone decreasing.
See definition in text!
Is connectivity monotone increasing, monotone decreasing, both or neither?
monotone increasing only: Added edges will not disconnect the graph; subtracted edges might.
Is Hamiltonicity monotone increasing, monotone decreasing, both or neither?
monotone increasing only: Once a Hamiltonian cycle exists, additional edges will not change that fact; removing edges might.
Is the existence of a legal 5 coloring of a graph monotone increasing, monotone decreasing, both or neither?
monotone decreasing only: Keep in mind that the property in question is not chromatic number! If a legal 5 coloring exists, removing edges cannot make the coloring illegal; adding in edges might.