Math 3322 Test II

DeMaio Spring 2009

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (20 points) Jason's CD collection consists of five different rock CD's, three different jazz CD's, two different blues CD's, two different classical CD's and a single folk CD. Jason is planning a trip and randomly selects four CD's.
   i. How many different ways can this be done? $\binom{13}{4} = 715$.
   ii. How many different ways can this be done if the folk CD must be one of the four? $\binom{12}{3} = 220$
   iii. How many different ways can this be done if Jason will take exactly one rock CD? $5\binom{8}{3} = 280$
   iv. How many different ways can this be done if Jason will take at least one rock CD? $\binom{13}{4} - \binom{8}{4} = 645$

2. (5 points) A freshman member of a student organization is sent to purchase two dozen donuts for the next club meeting. An ample supply of each of the donut types (chocolate icing, vanilla icing, jelly-filled, creme-filled and plain) is available. How many different ways can the freshman purchase the two dozen donuts? $\binom{5+24-1}{24} = 20\,475$

3. (10 points) Dr. Jones teaches a class of fifteen archeology students.
   i. How many different ways can students earn the five different possible letter grades? $5^{15} = 30\,517\,578\,125$
   ii. If the grade distribution will have 2 A's, 5 B's, 5 C's, 2 D's and a single F, how many different ways can the grades be assigned? $\binom{15}{2}\binom{13}{5}\binom{8}{5}\binom{3}{2}\binom{1}{1} = 22\,702\,680$

4. (10 points) How many different arrangements exist of the letters in the element
   i. gold; $4! = 24$
   ii. mercury; $\frac{7!}{2!} = 2520$
   iii. manganese? $\frac{9!}{2!2!2!} = 45\,360$

5. (5 points) Every day Don lunches at a restaurant that offers a lunch combo with a choice of beverage, entrée and side. If six drinks, twelve entrées and eight sides are available, on what day will Don first be forced to repeat a combo order? $6 * 12 * 8 + 1 = 577$

6. (5 points) Shot in the back while playing poker, Wild Bill Hickok's final hand was a pair of aces and a pair of eights, now known as the dead man's hand. How many different dead man's hands exist? $\binom{4}{2}^2 * 44 = 1584$

7. (10 points) Compute the probability of a *pair* in the game of poker. Note that $13\binom{4}{2}\binom{12}{3}4^3 = 1\,098\,240$ different pairs exist. Thus, $P(pair) = \frac{13\binom{4}{2}\binom{12}{3}4^3}{\binom{52}{5}} = \frac{352}{833} = 0.422\,57$.

8. (5 points) When set to random shuffle, Frank's compact disk player will play a random sequencing of all tracks from one CD without repetition. On a particular CD with fifteen different tracks, Frank has four favorite tracks. What is the probability that all four favorite tracks will be the first four tracks played? $\frac{4!*11!}{15!} = \frac{958\,003\,200}{15!} = \frac{1}{1365} = 7.326 \times 10^{-4}$

9. (5 points) In a certain class there are five seniors, seven juniors, eight sophomores and two freshman. When selecting three students at random, what is the probability that they are all the same year? $p = \frac{\binom{5}{3}+\binom{7}{3}+\binom{8}{3}+\binom{2}{3}}{\binom{22}{3}} = \frac{101}{1540} = 6.558\,4 \times 10^{-2}$

10. (10 points) Susan buys an economy pack of fifty pens. The pens are identical except for color. There are ten of each of five different colors.
    i. How many different ways can Susan select four pens of different colors to take to work? $\binom{5}{4} = 5$
    ii. How many different ways can Susan select ten pens to take to work? $\binom{5+10-1}{10} = 1001$
    iii. How many different ways can Susan select ten pens to take to work with at least one of each color? $\binom{5+5-1}{5} = 126$

11. (10 points) Use a combinatorial proof to show $\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn$ for $m, n \in \mathbb{Z}^+$.
Let $A = \{1, 2, 3, ..., m-1, m, m+1, ..., m+n\}$ and let S be the collection of all two element subsets of A. One the one hand, $|S| = \binom{m+n}{2}$. On the other hand partition A into $B = \{1, 2, 3, ..., m\}$ and $C = \{m+1, m+2, ...m+n\}$. Note that $|B| = m$ while $|C| = n$. How can we select two elements of A relative to B and C? We can select two elements from B in $\binom{m}{2}$ ways or we can select two elements from C in $\binom{n}{2}$ or we can select one element from B and one element from C in mn ways. Thus, $|S| = \binom{m}{2} + \binom{n}{2} + mn$. This shows that $\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn$ for $m, n \in \mathbb{Z}^+$.

12. (10 points) Use a combinatorial proof to show $\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$ for integers $j \leq k \leq n$.
Let $A = \{1, 2, ..., n\}$ and let S be the collection of all ordered pairs (B, C) such that $C \subseteq B \subseteq A$, $|B| = k$ and $|C| = j$. One the one hand first select B and then pick C from B. Selecting a subset of size k from A can be done in $\binom{n}{k}$ ways. Now select C from B. This can be done in $\binom{k}{j}$ ways. Thus, $|S| = \binom{n}{k}\binom{k}{j}$. On the other hand, pick C and then build B around C. Selecting a subset of size j from A can be done in $\binom{n}{j}$ ways. Now that C is established, we know j of the elements of B. We only need $k - j$ additional elements. Furthermore, we cannot repeat those selections of elements of C from A. We have only $n - j$ elements to consider for B. So we can complete B from C in $\binom{n-j}{k-j}$ ways. Hence, $|S| = \binom{n}{j}\binom{n-j}{k-j}$. Put the two together and $\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$ for integers $j \leq k \leq n$.

13. (10 points) At a particular university, a student's password consists of five lowercase letters such that no letters are repeated, the first letter is a letter from the student's first name and the last letter is any vowel (no y's allowed), not necessarily from the student's first name. How many different passwords can Mark create?
Pick the last letter first. The problem here is that we don't know if the last letter is an 'a' or another vowel. So, there are two disjoint cases: the last letter is an 'a' or the last letter is not an 'a.' Pick the last letter first, then the first from Marks' name and then the rest in order. By the sum rule this can be done in $1 * 3 * 24 * 23 * 22 + 4 * 4 * 24 * 23 * 22 = 230\,736$