Math 3322 Test II
DeMaio Summer 2009

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (15 points) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
   i. How many permutations of the elements in A exist? $8! = 40\,320$
   ii. How many 3-permutations of the elements in A exist? $\frac{8!}{(8-3)!} = 336$
   iii. How many subsets of size 3 of A exist? $\binom{8}{3} = 56$

2. (10 points) State the Binomial Theorem. For all $n \in \mathbb{Z}^+$, $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$.
   Use the Binomial Theorem to expand $(2x-3)^4$. You must show all the details of your work.
   $(2x-3)^4 = \binom{4}{4}(2x)^4(-3)^0 + \binom{4}{3}(2x)^3(-3)^1 + \binom{4}{2}(2x)^2(-3)^2 + \binom{4}{1}(2x)^1(-3)^3 + \binom{4}{0}(2x)^0(-3)^4$
   $= 16x^4 - 4*8*3x^3 + 6*4*9x^2 - 4*2*27x + 81 = 16x^4 - 96x^3 + 216x^2 - 216x + 81$

3. (10 points) Find the coefficient of x^{12} in the expansion of
   i. $(x-3)^{15}$; $\binom{15}{12}x^{12}(-3)^3 = -12\,285x^{12}$
   ii. $\left(3x^2-2\right)^8$; $\binom{8}{6}\left(3x^2\right)^6(-2)^2 = 81\,648x^{12}$
   iii. $\left(4x^5+3\right)^9$. 0 or does not exist since $5i \neq 12$ for any integer i.

4. (10 points) Provide a combinatorial proof that $\binom{3n}{2} = 3\binom{n}{2} + 3n^2$ for $n \in \mathbb{Z}^+$.
   Let $A = \{1, 2, ..., 3n\}$ and let S be the collection of all 2 element subsets of A. On one hand, $|S| = \binom{3n}{2}$. On the other hand, partition A into three subsets: $B = \{1, 2, ..., n\}$, $C = \{n+1, n+2, ..., 2n\}$ and $D = \{2n+1, 2n+2, ...3n\}$. How can we select two elements from A relative to B, C and D? We can select two elements from the same subset or we can select two subsets and pick one element from each. We can select two elements from the same subset by first selecting the subset in 3 ways and then selecting two elements from the subset in $\binom{n}{2}$ ways. Or we can select two subsets in $\binom{3}{2} = 3$ ways and pick one element from each in $n*n = n^2$ ways. Thus, $|S| = 3\binom{n}{2} + 3n^2$. We have counted the same set in two different ways and $\binom{3n}{2} = 3\binom{n}{2} + 3n^2$ for $n \in \mathbb{Z}^+$.

5. (10 points) Provide an algebraic proof that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
   $\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} + \frac{(n-1)!}{k!(n-1-k)!} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!}$
   $= \frac{k*(n-1)!+(n-k)*(n-1)!}{k!(n-k)!} = \frac{(k+n-k)*(n-1)!}{k!(n-k)!} = \frac{n*(n-1)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$.

6. (5 points) What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state? $50*99+1 = 4951$

7. (5 points) In our twenty person club, we have eight seniors, four juniors, three sophomores and five freshmen. How man ways can we select a five person committee with at least one senior? $\binom{20}{5} - \binom{12}{5} = 14\,712$.

8. (20 points) A social club consists of 10 pairs of twins, some identical, some fraternal. Of these 20 members, 13 are women and 7 are men.
   i. How many ways can this club select a social coordinator and treasurer if these roles must be served by different club members? $20*19 = 380$
   ii. How many ways can this club select a social coordinator and treasurer if these roles must be served by club members of opposite gender? $2*13*7 = 182$
   iii. How many ways can this club form a six person committee to coordinate the club trip to the annual national meeting of twins? Note that each person on this committee is of equal rank and power. $\binom{20}{6} = 38\,760$
   iv. How many ways can this club form a six person committee that consists of three twin pairs to coordinate the club trip to the annual national meeting of twins? Note that each person on this

committee is of equal rank and power. $\binom{10}{3} = 120$

v. How many ways can this club form a six person committee that consists of six club members but no twin pairs to coordinate the club trip to the annual national meeting of twins? Note that each person on this committee is of equal rank and power. $\binom{10}{6} * 2^6 = 13\,440$

9. (10 points) At a particular university, a student's password consists of five lowercase letters such that no letters are repeated, the first letter is a letter from the student's first name and the last letter is any vowel (no y's allowed), not necessarily from the student's first name. How many different passwords can Mark create?
   Pick the last letter first. The problem here is that we don't know if the last letter is an 'a' or another vowel. So, there are two disjoint cases: the last letter is an 'a' or the last letter is not an 'a.' Pick the last letter first, then the first from Marks' name and then the rest in order. By the sum rule this can be done in $1 * 3 * 24 * 23 * 22 + 4 * 4 * 24 * 23 * 22 = 230\,736$

10. (10 points) A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels and plain bagels. How many ways are there to choose
    i. 6 bagels; $n = 8, k = 6$ unordered set with repetition: $\binom{8+6-1}{6} = 1716$
    ii. 2 dozen bagels; $n = 8, k = 24$ unordered set with repetition: $\binom{8+24-1}{24} = 2629\,575$
    iii. a dozen bagels with at least one of each kind? If we pick one of each flavor to start we really only make a selection of 4 additional bagels. $n = 8, k = 4$ unordered set with repetition: $\binom{8+4-1}{4} = 330$

11. (5 points) Prove $\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$ for $n \in \mathbb{Z}^+$.

    In the binomial theorem which states $(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$, let $x = -1$ and $y = 1$. Thus, $\sum_{i=0}^{n} \binom{n}{i}(-1)^i = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \, 1^{n-i} = (-1 + 1)^n = 0^n = 0$.

12. (5 points) True or False: $\binom{n}{j+k} = \binom{n}{j} + \binom{n}{k}$ for all $n, j, k \in \mathbb{Z}^+$. If true, prove it. If false, provide a counterexample. False! Let $n = 10$ and $j = k = 1$. Now $\binom{n}{j+k} = \binom{10}{2} = 45$ while $\binom{n}{j} + \binom{n}{k} = \binom{10}{1} + \binom{10}{1} = 20$.