Math 3322 Test II
DeMaio Spring 2013

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. Where appropriate, simplify all answers to a single decimal expansion.

1. (10 points) Let $g_0 = 1$. Let $g_n = 1 - 2g_{n-1}$ for $n \geq 1$. Compute g_1, g_2, g_3 and g_4.
   $g_1 = 1 - 2 * 1 = -1$
   $g_2 = 1 - 2(-1) = 3$
   $g_3 = 1 - 2 * 3 = -5$
   $g_4 = 1 - 2(-5) = 11$

2. (10 points) Give a recursive definition of the set of positive integers that are not divisible by 5.
   Let $a_n = a_{n-4} + 5$ for $n \geq 5$ where $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_4 = 4$.

3. (10 points) Complete the table of Fibonacci numbers.

n	0	1	2	3	4	5	6	7	8	9	10
f_n	0	1	1	2	3	5	8	13	21	34	55

4. (15 points) Use induction to prove $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$ for the Fibonacci sequence and $n \in Z^+$.First show that the base case of $n = 1$ hold true.
   R.H.S. $\sum_{i=1}^{1} F_i^2 = F_1^2 = 1^2 = 1$ while L.H.S. $F_1 F_{1+1} = F_1 F_2 = 1 * 1 = 1$
   Assume the statement is true for $n : \sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$. Show that the statement is true for the value $n + 1 : \sum_{i=1}^{n+1} F_i^2 = F_{n+1} F_{n+2}$.
   Note that $\sum_{i=1}^{n+1} F_i^2 = \sum_{i=1}^{n} F_i^2 + F_{n+1}^2$ which by the inductive hypothesis is $F_n F_{n+1} + F_{n+1}^2 = F_{n+1}(F_n + F_{n+1}) = F_{n+1} F_{n+2}$.

5. (15 points) Use induction to show $\frac{2^{4n}-61}{15}$ is an integer for $n \in \mathbb{Z}^+$.
   First, show $S(1)$ is true. Note, $\frac{2^{4*1}-61}{15} = -3$ which is an integer.
   Assume $\frac{2^{4n}-61}{15}$ is an integer and show $\frac{2^{4(n+1)}-61}{15} = \frac{2^{4n+4}-61}{15}$ is an integer.
   Well, $\frac{2^{4n+4}-61}{15} = \frac{2^4 2^{4n}-61}{15} = \frac{2^{4n}-61}{15} + \frac{15*2^{4n}}{15} = \frac{2^{4n}-61}{15} + 2^{4n}$. The first part of the sum is an integer by the inductive hypothesis and 2^{4n} is clearly an integer for all integers n. The sum of two integers is an integer. Thus, $\frac{2^{4n+4}-61}{15}$ is an integer.

6. (5 points) A theater concession counter offers four different sizes of drinks and eight different choices of beverages. How many different ways can a drink be ordered?$4 * 8 = 32$

7. (5 points) A restaurant offers a daily lunch special. This special consists of a salad with a choice of ten dressings, five different sandwiches, three different desserts and a drink choice consisting of water, tea, Coke or diet Coke. If you must pick one each of salad dressing, sandwich, dessert and beverage, how many different lunch specials may be ordered?$10 * 5 * 3 * 4 = 600$

8. (10 points) How many different passwords of seven characters exist where each character is a lowercase letter of the alphabet and each password contains at least one vowel and at least one consonant?$26^7 - 21^7 - 5^7 = 6230\,643\,510$

9. (10 points) How many positive integers not exceeding 10,000 are divisible by 6 or 15?$\lfloor \frac{10000}{6} \rfloor + \lfloor \frac{10000}{15} \rfloor - \left\lfloor \frac{10000}{\text{lcm}(6,15)} \right\rfloor = 1999$

10. (15 points) A collection of seven distinct coins will be arranged from left to right. There are four heads face up and three tails face up.

    i. How many different ways can the coins be arranged from left to right? $7! = 5040$

    ii. How many different ways can the coins be arranged from left to right if there can be no consecutive heads? $4 * 3 * 3 * 2 * 2 * 1 * 1 = 144$

    iii. How many different ways can the coins be arranged from left to right if all heads must be consecutive and all tails must be consecutive? $2 * 4! * 3! = 288$

11. (10 points) What is the minimum number of students, each of whom comes from one of the fifty states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state? $50 * 99 + 1 = 4951$