Math 3322 Test II
DeMaio Fall 2012

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (30 points) In the questions below pick a sequence of five cards (without replacement) and arrange them face up in a sequence from left to right. Recall that there are four suits: clubs, diamonds, hearts spades and 13 ranks: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K. There is exactly one card of each rank and suit for a total of 52 cards.
   i. How many sequences are possible? $52 * 51 * 50 * 49 * 48 = 311\,875\,200$
   ii. How many of these sequences have no repeated ranks? $52 * 48 * 44 * 40 * 36 = 158\,146\,560$

   iii. How many of these sequences begin with a King or end in an Ace? $4 * 51 * 50 * 49 * 48 + 4 * 51 * 50 * 49 * 48 - 4 * 4 * 50 * 49 * 48 = 46\,099\,200$

   iv. How many of these sequences have no repeated suits? 0

   v. How many of these sequences begin with a King or a club?$16 * 51 * 50 * 49 * 48 = 95\,961\,600$

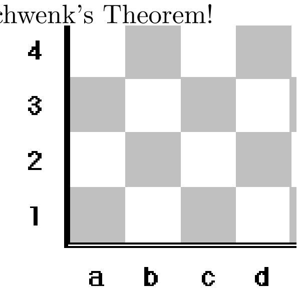
   vi. How many sequences have no clubs? $39 * 38 * 37 * 36 * 35 = 69\,090\,840$

2. (10 points) Write a recursive definition of the positive multiples of 7. Let $S_n = S_{n-1} + 7$ for $n \geq 2$ where $S_1 = 7$.

3. (10 points) Let $f(0) = 3$ and $f(n) = \left\lfloor \frac{14}{f(n-1)} \right\rfloor$ for $n \geq 1$. Compute the following.
   i. $f(1) = \lfloor \frac{14}{3} \rfloor = 4$
   ii. $f(2) = \lfloor \frac{14}{4} \rfloor = 3$
   iii. $f(3) = \lfloor \frac{14}{3} \rfloor = 4$
   iv. $f(4) = \lfloor \frac{14}{4} \rfloor = 3$

4. (15 points) Prove that any amount of postage ≥ 18 cents can be made using only 4 cent and/or 7 cent stamps.
   $18 = 2 * 7 + 4$
   $19 = 7 + 3 * 4$
   $20 = 5 * 4$
   $21 = 3 * 7$
   Assume that any amount of postage $18, 19, 20, 21, ..., n$ cents for $n \geq 21$ can be made using only 4 cent and/or 7 cent stamps. Show how to make $n + 1$ cents of postage.
   $n + 1 = 4 + n - 3$
   Note that $n - 3 \leq n$ and since $n \geq 21$ then $n - 3 \geq 18$.
   Thus, by inductive assumption we can make $n - 3$ cents worth of postage using only 4 cent and/or 7 cent stamps. Add a 4 cent stamp to that solution and we've made $n + 1$ cents of postage using only 4 cent and/or 7 cent stamps.

5. (15 points) Prove that the 4×4 board does not admit a closed knight's tour. You may not reference Schwenk's Theorem!



6. (15 points) Use the fact that the $\gcd(F_n, F_{n+1}) = 1$ to prove $\gcd(F_n, F_{n+2}) = 1$.
   Assume $\gcd(F_n, F_{n+2}) = d > 1$ for some n. Since $F_{n+2} = F_n + F_{n+1}$ then $F_{n+1} = F_{n+2} - F_n$. Since

$\gcd(F_n, F_{n+2}) = d$ then d divides both F_n and F_{n+2}. Thus, d divides their difference, F_{n+1}. However, now d divides both F_n and F_{n+1} so $1 = \gcd(F_n, F_{n+2}) \geq d > 1$ which is a problem. Hence, the assumption is wrong and $\gcd(F_n, F_{n+2}) = 1$.

7. (15 points) Determine a formula for $\sum_{i=1}^{n} F_{2i}$. Use induction to prove the correctness of your conjecture.

$F_2 = 1$
$F_2 + F_4 = 1 + 3 = 4$
$F_2 + F_4 + F_6 = 1 + 3 + 8 = 12$.
Conjecture: $\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$.

Assume the statement is true for $n : \sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$. Show that the statement is true for the value $n+1 : \sum_{i=1}^{n+1} F_{2i} = F_{2(n+1)+1} - 1 = F_{2n+3} - 1$.
$\sum_{i=1}^{n+1} F_{2i} = \sum_{i=1}^{n} F_{2i} + F_{2n+2}$ which by the inductive hypothesis is $F_{2n+1} - 1 + F_{2n+2} = F_{2n+3} - 1$.