Math 3322 Test III
DeMaio Fall 2011

Name______________________________

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (15 points) State the Binomial Theorem. For any x and y, $(x+y)^n = \sum_{i=0}^{n} x^i y^{n-i} \binom{n}{i}$.

   Use the Binomial Theorem to expand $(2x-3)^4$. You must show all the details of your work.
   $(2x-3)^4 = 16x^4 - 96x^3 + 216x^2 - 216x + 81$

2. (15 points) Find the coefficient of x^{12} in the expansion of
   i. $(x-3)^{15}$; $\binom{15}{12}(-3)^3 = -12\,285$
   ii. $\left(3x^2-2\right)^8$; $3^6(-2)^2\binom{8}{6} = 81\,648$
   iii. $\left(4x^5+3\right)^9$. 0 or does not exist

3. (5 points) Express $1296x^{12} - 4320x^9y^2 + 5400x^6y^4 - 3000x^3y^6 + 625y^8$ in the form $(a+b)^n$. Note that powers of x are multiples of 3 and powers of y are multiples of 2. So, we can start with $(?x^3 + ?y^2)^n$. From here it is obvious that $n = 4$. So the coefficient in fromt of the y term must be the fourth root of 625 and $625^{\frac{1}{4}} = 5$. Likewise, $1296^{\frac{1}{4}} = 6$. Finally note that the coefficient of y^2 is negative for odd powers. So, $\left(6x^3 - 5y^2\right)^4 = 1296x^{12} - 4320x^9y^2 + 5400x^6y^4 - 3000x^3y^6 + 625y^8$. Since the power is even $\left(5y^2 - 6x^3\right)^4 = 1296x^{12} - 4320x^9y^2 + 5400x^6y^4 - 3000x^3y^6 + 625y^8$ also works.

4. (10 points) Provide a combinatorial proof that $\binom{3n}{2} = 3\binom{n}{2} + 3n^2$ for $n \in \mathbb{Z}^+$.
   Let $A = \{1, ..., 3n\}$ and let S be the collection of all two-element subsets of A. On the one hand it is clear that $|S| = \binom{3n}{2}$. On the other hand, partition A into $B = \{1, 2, ..., n\}$, $C = \{n+1, n+2, ..., 2n\}$ and $D = \{2n+1, 2n+2, ..., 3n\}$. How can we select two elements from A relative to B, C and D? We can select a subset and pick two elements from that subset in $3\binom{n}{2}$ ways. Or we can select two subsets and pick one element from each in $\binom{3}{2}n * n = 3n^2$ ways. Thus, $|S| = 3\binom{n}{2} + 3n^2$. We;ve counted the same set in two different ways and $\binom{3n}{2} = 3\binom{n}{2} + 3n^2$ for $n \in \mathbb{Z}^+$.

5. (15 points) Carefully describe the sets S_n for $n \in \mathbb{Z}^+$ which are used to combinatorially interpret the Fibonacci sequence. Be certain to provide the identity for $|S_n|$.
   Use these sets to provide a combinatorial proof $F_{m+n} = F_{m+1}F_{n+1} - F_{m-1}F_{n-1}$.
   See clas notes.

6. (20 points) Three cards are randomly select from a standard deck. Compute the following probabilities.
   i. The probability that all three cards are Aces. $p = \frac{\binom{4}{3}}{\binom{52}{3}} = \frac{1}{5525} = 1.8100 \times 10^{-4}$
   ii. The probability that all three cards are the same suit. $p = \frac{4\binom{13}{3}}{\binom{52}{3}} = \frac{22}{425} = 5.1765 \times 10^{-2}$
   iii. The probability that at least one face card (Jack, Queen, King) is selected. $p = \frac{\binom{52}{3} - \binom{40}{3}}{\binom{52}{3}} = \frac{47}{85} = 0.55294$
   iv. The probability that all suits are different. $\frac{\binom{4}{3}13^3}{\binom{52}{3}} = \frac{169}{425} = 0.39765$

7. (20 points) A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, cinnamon bagels and plain bagels. How many ways are there to choose
   i. 5 different flavored bagels; $\binom{9}{5} = 126$
   ii. 2 dozen bagels; $n = 9, k = 24$ so $\binom{9+24-1}{24} = 10\,518\,300$
   iii. a dozen bagels with at least one of each flavor? $n = 9, k = 3$ so $\binom{9+3-1}{3}$: 165
   iv. two dozen bagels with at least one of each flavor and at most three plain bagels. We first select one of each flavor of bagel which leaves us with 15 more selections. Since we can have at most 3 plain bagels we can select at most 2 additional plain bagels. So we have three cases: 0 additional plain, 1 additional plain or 2 additional plain bagels. $\binom{8+15-1}{15} + \binom{8+14-1}{14} + \binom{8+13-1}{13} = 364\,344$

8. (5 points) For $x, y, z \in \mathbb{Z}^+$,how many different solutions exist to $x + y + z = 23$? $\binom{3+20-1}{20} = 231$

9. (10 points) Use the binomial theorem to determine x if $\sum_{i=0}^{60} 15^i \binom{60}{i} = x^{240}$. Show all details.

   By the binomial theorem $\sum_{i=0}^{60} 15^i \binom{60}{i} = \sum_{i=0}^{60} 1^{60-i} 15^i \binom{60}{i} = (1+15)^{60} = 16^{60}$. Now set $16^{60} = x^{240}$ and note that $x^{240} = \left(x^4\right)^{60}$. So $x^4 = 16$ and $x = \pm 2$.