Math 3322 Test III
DeMaio Spring 2013

Name______________________________________________

**Instructions.** Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (10 points) How many ways can we arrange the letters in the word
   i. zombie; $6! = 720$
   ii. werewolf? $\frac{8!}{2!2!} = 10\,080$

2. (5 points) How many solutions exist to $w + x + y + z = 23$ where $w, x, y$ and $z$ are positive integers and $z \leq 5$?
   $\binom{4+19-1}{19} - \binom{4+14-1}{14} = 860$

3. (5 points) How many functions exist from a domain with 5 elements to a codomain with 8 elements?
   $8^5 = 32\,768$

4. (5 points) How many one-to-one functions exist from a domain with 5 elements to a codomain with 8 elements?
   $\frac{8!}{3!} = 6720$

5. (10 points) How many onto functions exist from a domain with 11 elements to a codomain with 10 elements? $\binom{11}{2}10! = 199\,584\,000$

6. (15 points) Show all your work and use the binomial theorem to find the coefficient of $x^9$ in the expansion of
   i. $(x-2)^{10}$ ; $\binom{10}{9}x^9(-2)^1 = -20x^9$
   ii. $\left(3x^2-1\right)^7$ ; 0 or d.n.e.
   iii. $\left(4x^3+3\right)^9$ . $\binom{9}{3}\left(4x^3\right)^3(3)^6 = 3919\,104x^9$

7. (10 points) Prove $\sum_{i=0}^{n}\binom{n}{i} = 2^n$.

   Let $x = y = 1$ in the binomial theorem Thus, $\sum_{i=0}^{n}\binom{n}{i} = \sum_{i=0}^{n}\binom{n}{i}1^i1^{n-i} = (1+1)^n = 2^n$.

8. (20 points) Susan buys an economy pack of 100 pens from Costco. The pack contains ten identical pens of ten different colors. How many ways can Susan select
   i. eight pens of different colors; $\binom{10}{8} = 45$
   ii. eight pens; $\binom{10+8-1}{8} = 24\,310$
   iii. eight pens with the same number of each selected color;
   one color or two colors or four colors or eight colors.
   $\binom{10}{1} + \binom{10}{2} + \binom{10}{4} + \binom{10}{8} = 310$
   iv. twelve pens? $\binom{10+12-1}{12} - 10 - 10 * 9 = 293\,830$

9. (15 points) True or False? For all $n, k \in \mathbb{Z}^+$, $\binom{kn}{2} = k\binom{n}{2}$ for $k, n \in \mathbb{Z}^+$. If true, prove it. If false, provide a counterexample to show that the statement is not true.
   False: Let $k = n = 3$. $\binom{3*3}{2} = 36$ but $3\binom{3}{2} = 9$.

10. (10 points) Provide a combinatorial proof that $\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn$ for $m, n \in \mathbb{Z}^+$.
    Let $A = \{1, 2, 3, ..., m-1, m\}$ and $B = \{m+1, m+2..., m+n\}$. Note that $|A| = m$, $|B| = n$ and $|A \cup B| = m + n$. Suppose $S$ is the collection of all subsets of $A \cup B$ of size 2. Clearly, $|S| = \binom{m+n}{2}$. How else might we select a subset of size 2 from $A \cup B$? We could select two elements from $A$ in $\binom{m}{2}$ ways. Or, we could select two elements from $B$ in $\binom{n}{2}$ ways. Or we could select one element from $A$ in $m$ ways and select one element from $B$ in $n$ ways yielding $mn$ ways to select one eleemnt from each set. Thus, $|S| = \binom{m}{2} + \binom{n}{2} + mn$. We've counted the same set in two different ways and $\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn$.

11. (10 points) Provide an algebraic proof that $\binom{n}{k} = \binom{n}{n-k}$ for $n, k \in \mathbb{Z}^+$ where $k \leq n$.
$\binom{n}{n-k} = \frac{n!}{(n-k)![n-(n-k)]!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$