Math 3322 Test III
DeMaio Spring 2012

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (15 points) Use the binomial theorem to find the coefficient of x^{12} in the expansion of
   i. $(x-3)^{15}$; $\binom{15}{12}(-3)^3x^{12} = -12\,285x^{12}$
   ii. $\left(3x^2-2\right)^8$; $\binom{8}{6}\left(3x^2\right)^6(-2)^2 = 81\,648x^{12}$
   iii. $\left(4x^5+3\right)^9$. 0 (or does not exist)

2. (10 points) Use the binomial theorem to compute $\sum_{i=0}^{15} 2^i\binom{15}{i}$.
   $\sum_{i=0}^{15} 2^i\binom{15}{i} = \sum_{i=0}^{15} 2^i 1^{n-i}\binom{15}{i} = (1+2)^{15} = 14\,348\,907$

3. (10 points) Provide a combinatorial proof that $\binom{3n}{2} = 3\binom{n}{2} + 3n^2$ for $n \in \mathbb{Z}^+$.
   Let $A = \{1, ..., 3n\}$ and let S be the collection of all two-element subsets of A. On the one hand it is clear that $|S| = \binom{3n}{2}$. On the other hand, partition A into $B = \{1, 2, ..., n\}$, $C = \{n+1, n+2, ..., 2n\}$ amd $D = \{2n+1, 2n+2, ..., 3n\}$. How can we select two elements from A relative to B, C and D? We can select a subset and pick two elements from that subset in $3\binom{n}{2}$ ways. Or we can select two subsets and pick one element from each in $\binom{3}{2}n * n = 3n^2$ ways. Thus, $|S| = 3\binom{n}{2} + 3n^2$. We;ve counted the same set in two different ways and $\binom{3n}{2} = 3\binom{n}{2} + 3n^2$ for $n \in \mathbb{Z}^+$.

4. (30 points) Recall our 20 person club with 13 women and 7 men. How many ways can we
   i. elect a President and Vice-President; $20 * 19 = 380$
   ii. form a committee of three people (where each member has equal rank and power); $\binom{20}{3} = 1140$
   iii. form a committee of four people and elect one of those four members to be the chairperson; $\binom{20}{4} * 4 = 19\,380$
   iv. form a committee of two people of opposite gender; $13 * 7 = 91$
   v. form a committee of two people of the same gender; $\binom{20}{2} - 91 = 99$ or $\binom{13}{2} + \binom{7}{2} = 99$
   vi. form a committee of six people with at least one member of each gender? $\binom{20}{6} - \binom{13}{6} - \binom{7}{6} = 37\,037$

5. (10 points) How many ways can we rearrange the letters in the word
   i. vampire; $7! = 5040$
   ii. werewolf; $\frac{8!}{2!2!} = 10\,080$

6. (10 points) How many positive integers not exceeding 10,000 are divisible by 6 or 15? $\left\lfloor \frac{10000}{6} \right\rfloor + \left\lfloor \frac{10000}{15} \right\rfloor - \left\lfloor \frac{10000}{\text{lcm}(6,15)} \right\rfloor = 1999$

7. (15 points) A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, cinnamon bagels and plain bagels. How many ways
   i. can Mark choose 5 different flavored bagels; $\binom{9}{5} = 126$
   ii. can Mark choose a dozen bagels and Karen choose a dozen bagels; $\binom{9+12-1}{12}^2 = 15\,868\,440\,900$
   iii. can Mark choose three dozen bagels with at least two of each flavor? $\binom{9+18-1}{18} = 1562\,275$

8. (5 points) How many functions exist from a domain with 7 elements to a codomain with 10 elements? $10^7 = 10\,000\,000$

9. (5 points) For $w, x, y, z \in \mathbb{Z}$, $w \geq 1, x \geq 2, y \geq 3$ and $z = 5$, how many different solutions exist to $w + x + y + z = 35$? $\binom{3+24-1}{24} = 325$

10. (5 points) What is the minimum number of students, each of whom comes from one of the fifty states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state? $50 * 99 + 1 = 4951$