Math 3322 Test III
DeMaio Fall 2012

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (15 points) How many ways can we arrange the letters in the word
   i. vampire; $7! = 5040$
   ii. Buffy; $\frac{5!}{2!} = 60$
   iii. werewolf. $\frac{8!}{2!2!} = 10\,080$

2. (5 points) How many functions exist from a domain with 7 elements to a codomain with 10 elements? $10^7 = 10\,000\,000$

3. (5 points) How many one-to-one functions exist from a domain with 7 elements to a codomain with 10 elements? $10 * 9 * 8 * 7 * 6 * 5 * 4 = 604\,800$

4. (10 points) How many onto functions exist from a domain with 7 elements to a codomain with 6 elements? $\binom{7}{2} * 6! = 15\,120$

5. (15 points) Show all your work and use the binomial theorem to find the coefficient of x^8 in the expansion of
   i. $(x-2)^{10}$; $\binom{10}{8} x^8 (-2)^2 = 180x^8$
   ii. $\left(3x^2 - 1\right)^7$; $\binom{7}{4} \left(3x^2\right)^4 (-1)^3 = -2835x^8$
   iii. $\left(4x^3 + 3\right)^9$. 0 (or does not exist)

6. (10 points) Use the binomial theorem to compute $\sum_{i=0}^{15} (-3)^i \binom{15}{i}$.
   $\sum_{i=0}^{15} (-3)^i \binom{15}{i} = \sum_{i=0}^{15} (-3)^i \binom{15}{i} * 1^{n-i}$ which by the binomial theorem is $(1-3)^{15} = (-2)^{15} = -32\,768$

7. (10 points) A drawer contain 12 black socks, 10 blue socks and 8 brown socks. A man takes out socks at random in the dark.
   i. How many socks must he take out to be sure that he has at least two socks of the same color? There are three pigeonholes which are labeled with the different sock colors. The pigeons are the socks. He must remove 4 socks since $\left\lceil \frac{4}{3} \right\rceil = 2$ but $\left\lceil \frac{3}{3} \right\rceil = 1$.
   ii. How many socks must he take out to be sure that he has at least two blue socks? He must remove 22 socks since it is possible that his first $12 + 8 = 20$ selections were not blue.

8. (10 points) Recall our 20 person club with 13 women and 7 men. How many ways can we
   i. form a committee of three people (where each member has equal rank and power); $\binom{20}{3} = 1140$
   ii. form a committee of four people and elect one of those four members to be the chairperson; $\binom{20}{4} * 4 = 19\,380$ or $20 * \binom{19}{3} = 19\,380$.

9. (10 points) What is the minimum number of students, each of whom comes from one of the fifty states, who must be enrolled in a university to guarantee that there are at least 250 who come from the same state? $249 * 50 + 1 = 12\,451$

10. (15 points) True or False? For all $n, k \in \mathbb{Z}^+$, $\binom{n}{2} + \binom{k}{2} = \binom{n+k}{2}$. If true, prove it. If false, provide a counterexample to show that the statement is not true.
    False! Let $n = k = 2$.
    $\binom{2}{2} + \binom{2}{2} = 2$ but $\binom{2+2}{2} = 6$

11. (5 points) Provide a combinatorial proof that $\binom{n}{k} = \binom{n}{n-k}$ for $n, k \in \mathbb{Z}^+$ where $k \leq n$.
    A subset of an n element set of size k can be selected in $\binom{n}{k}$ ways. However, selecting that subset of size k is equivalent to selecting the $n-k$ elements that are not in the subset (the complement). That selection can be made in $\binom{n}{n-k}$ ways. Hence, $\binom{n}{k} = \binom{n}{n-k}$.

12. (5 points) Provide an algebraic proof that $\binom{n}{k} = \binom{n}{n-k}$ for $n, k \in \mathbb{Z}^+$ where $k \leq n$.

$\binom{n}{n-k} = \frac{n!}{(n-k)! * (n-(n-k))!} = \frac{n!}{(n-k)! * k!} = \binom{n}{k}$