Math 3322 Test III
DeMaio Spring 2011

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (45 points) In the questions below suppose that a "word" is any string of five uppercase letters of the alphabet, with repeated letters allowed. Furthermore, always classify the pesky "sometimes Y" as a consonant.
   i. How many words exist? $26^5 = 11\,881\,376$
   ii. How many words end with the letter B? $26^4 = 456\,976$
   iii. How many words begin with A and end with B? $26^3 = 17\,576$
   iv. How many words begin with A or B? $2 * 26^4 = 913\,952$
   v. How many words begin with A or end with B? $26^4 + 26^4 - 26^3 = 896\,376$
   vi. How many words have no vowels? $21^5 = 4084\,101$
   vii. How many words have exactly one vowel? $5 * 5 * 21^4 = 4862\,025$
   viii. How many words have at least one vowel? $26^5 - 21^5 = 7797\,275$
   ix. How many words have exactly two vowels? $\binom{5}{2} * 5^2 * 21^3 = 2315\,250$

2. (5 points) Consider a twenty person club. How many different ways can a President, Vice-President and Treasurer be elected?$20 * 19 * 18 = 6840$

3. (5 points) How many functions are there from a domain with 7 elements to a codomain with 10 elements? $10^7 = 10\,000\,000$

4. (5 points) How many one-to-one functions are there from a domain with 7 elements to a codomain with 10 elements? $\frac{10!}{(10-7)!} = 604\,800$

5. (5 points each) Let $S = \{1, 2, 3, 4, 5, 6, 7\}$.
   (a) How many non-empty subsets does S have? $2^7 - 1 = 127$
   (b) How many subsets of S have no odd numbers as members? $2^3 = 8$
   (c) How many subsets of S have exactly 4 elements? $\binom{7}{4} = 35$
   (d) How many subsets of S have at least 5 elements? $\binom{7}{5} + \binom{7}{6} + \binom{7}{7} = 29$
   (e) How many even four digit numbers can be made using the digits of S if a digit may be used only once? Pick the last digit first and then the first, second and third digits. This can be done in $3 * 6 * 5 * 4 = 360$ ways.

6. (10 points) Use sophisticated counting techniques to determine how many positive integers not exceeding 1,000 are divisible by 9 or 12? $\left\lfloor \frac{1000}{9} \right\rfloor + \left\lfloor \frac{1000}{12} \right\rfloor - \left\lfloor \frac{1000}{\text{lcm}(9,12)} \right\rfloor = 167$

7. (10 points) Use the Pigeonhole Principle to show that if seven **distinct** numbers are selected from $\{1, 2, ..., 11\}$, then some two of these numbers sum to 12. For an extra 5 **bonus points** instead show that if $n + 1$ numbers are selected from $\{1, 2, ..., 2n - 1\}$, then some two of these numbers sum to $2n$. In either case you must carefully describe the pigeonholes and how the pigeons are placed.
   The numbers $\{1, 2, ..., 2n-1\}$ are the $2n-1$ pigeons. The n pigeonholes are labeled with two numbers on each pigeonhole as $\{1, 2n-1\}, \{2, 2n-1\}, \{3, 2n-3\}, ..., \{n-1, n+1\}$ and a last pigeonhole labeled only as $\{n\}$. Now select $n+1$ entries from $\{1, 2, ..., 2n-1\}$ and place the numbers into their pigeonholes according to label. You've placed $n+1$ pigeons into n pigeonholes and at least one pigeonhole contains at least two pigeons. Since the selected numbers are distinct we know that some pigeonhole other than than the one labeled $\{n\}$ has two pigeons of distinct values. By the way we labeled the pigeonholes, we know that the sum of those two numbers is $2n$.

8. (10 points) Use the binomial coefficient to show why it is necessary to define $0! = 1$. Combinatorially we know that $\binom{n}{n} = 1$ since there is only one way to select the entire set from itself. Plugging into the formula for the binomial coefficient we see that $1 = \frac{n!}{n!0!}$. Canceling the $n!$s reveals that $1 = \frac{1}{0!}$. Solving for $0!$ gives us that $0! = 1$.