Math 3322 Test IV
DeMaio Fall 2012

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (20 points) A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, cinnamon bagels and plain bagels. How many ways are there to choose
   i. five different flavored bagels; $\binom{9}{5} = 126$
   ii. two dozen bagels; $\binom{9+24-1}{24} = 10\,518\,300$
   iii. two dozen bagels with at least one of each flavor; $\binom{9+(24-9)-1}{24-9} = \binom{9+15-1}{15} = 490\,314$
   iv. two dozen bagels with no plain bagels, at least three egg bagels and at most three pumpernickel bagels? $\binom{8+21-1}{21} - \binom{8+17-1}{17} = 837\,936$

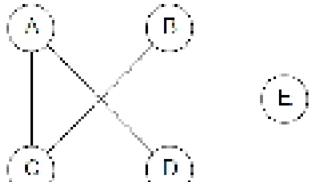
2. (20 points) How many different solutions exist to $x + y + z = 23$?
   i. for $x, y, z \in \mathbb{Z}^+ \cup \{0\}$; $\binom{3+23-1}{23} = 300$
   ii. for $x, y, z \in \mathbb{Z}^+ \cup \{0\}$ where $x \geq 3$ and $z \geq 6$; $\binom{3+(23-9)-1}{23-9} = \binom{3+14-1}{14} = 120$
   iii. for $x, y, z \in \mathbb{Z}^+$ where $x \geq 10$ and $z \leq 5$? $\binom{3+11-1}{11} - \binom{3+5-1}{5} = 57$
   iv. This problem changes dramatically if we allow $x, y, z \in \mathbb{Z}$? Explain why? Permitting negative integers will allow for an infinite number of solutions.

3. (20 points) Three cards are randomly select from a standard deck. Recall that there are four suits: clubs, diamonds, hearts spades and 13 ranks: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K. There is exactly one card of each rank and suit for a total of 52 cards.
   Compute the following probabilities.
   i. The probability that all three cards are Aces. $p = \frac{\binom{4}{3}}{\binom{52}{3}} = \frac{1}{5525} = 1.8100 \times 10^{-4}$
   ii. The probability that all three cards are the same suit. $p = \frac{4 * \binom{13}{3}}{\binom{52}{3}} = \frac{22}{425} = 5.1765 \times 10^{-2}$
   iii. The probability that at least one face card (Jack, Queen, King) is selected. $p = \frac{\binom{52}{3} - \binom{40}{3}}{\binom{52}{3}} = \frac{47}{85} = 0.55294$
   iv. The probability that all suits are different. $p = \frac{\binom{4}{3} * 13^3}{\binom{52}{3}} = \frac{169}{425} = 0.39765$

4. (10 points) How many positive integers not exceeding 1000 are divisible by 6, 9 or 15? $\left\lfloor \frac{1000}{6} \right\rfloor + \left\lfloor \frac{1000}{9} \right\rfloor + \left\lfloor \frac{1000}{15} \right\rfloor - \left\lfloor \frac{1000}{\text{lcm}(6,9)} \right\rfloor - \left\lfloor \frac{1000}{\text{lcm}(6,15)} \right\rfloor - \left\lfloor \frac{1000}{\text{lcm}(9,15)} \right\rfloor + \left\lfloor \frac{1000}{\text{lcm}(6,9,15)} \right\rfloor = 244$

5. (10 points) Can the following scenario occur? Explain. There are 95 students who play at least one of football, basketball and baseball. There are 64 football players, 28 basketball players and 29 baseball players. There are 17 students who play both football and basketball, 13 students who play both football and baseball and 12 students who play both basketball and baseball. Let x be the number of students who play all three sports. By inclusion/exclusion we know that $95 = 64 + 28 + 29 - 17 - 13 - 12 + x$, and $x = 16$. But this is a problem! It is illogical to have 16 students play all three sports when only 12 play both basketball and baseball. Thus, this scenario is not possible.

6. (10 points) Draw the intersection graph for sets $A = \{1, 4, 5, 8, 9\}$, $B = \{2, 4, 5, 6, 9, 10\}$, $C = \{1, 2, 3\}$, $D = \{1, 8, 9\}$ and $E = \{7\}$.



7. (10 points) If possible, draw a graph with nine vertices such that five of the vertices have degree 5 and the other four vertices have degree 4. If not possible, explain why such a graph does not exist. This graph does not exist. If it did, $\sum_{v \in V} \deg(v) = 5 * 5 + 4 * 4 = 41$. However, we know by the handshaking lemma that the sum of the degrees must be even.

8. (10 points) Let G be a graph with $n \geq 2$ vertices. Prove that G has at least two vertices u and v such that $\deg(u) = \deg(v)$.
With n vertices, $0 \leq \deg(v) \leq n - 1$ for every vertex $v \in V$. However, if $\deg(v) = n - 1$ then v is adjacent to every other vertex in the graph. This implies that no isolated vertices exist. Thus, vertex degrees of 0 and $n - 1$ cannot coexist in the same graph. Hence, there are $n - 1$ different possible vertex degrees that can exist in a graph. These possible degrees represent $n-1$ pigeonholes. We place n vertices (pigeons) into these pigeonholes based on degree. The pigeonhole principle states that at least one of the pigeonholes will contain at least two pigeons. Thus, there are two vertices that have the same degree.

9. (5 points) List the degree sequence of P_6. $2, 2, 2, 2, 1, 1$