Math 3322 Test IV
DeMaio Spring 2011

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. **Where appropriate, simplify all answers to a single decimal expansion.**

1. (15 points) Use the binomial theorem to expand $(3x-2)^4$ into polynomial form. You must show all details.
$(3x-2)^4 = \binom{4}{4}(3x)^4(-2)^{4-4} + \binom{4}{3}(3x)^3(-2)^{4-3} + \binom{4}{2}(3x)^2(-2)^{4-2} + \binom{4}{1}(3x)^1(-2)^{4-1} + \binom{4}{0}(3x)^0(-2)^{4-0} = 81x^4 - 216x^3 + 216x^2 - 96x + 16$

2. (10 points each) Find the coefficient of x^{10} in the expansion of
(a) $(2x-4)^{15}$; $\binom{15}{10}(2x)^{10}(-4)^5 = -3148\,873\,728x^{10}$
(b) $\left(4x^3-3\right)^{12}$; The coefficient is 0 since no integer power of $4x^3$ will yield x^{10}.
(c) $\left(8x^5-3\right)^5$. $\binom{5}{2}\left(8x^5\right)^2(-3)^3 = -17\,280x^{10}$

3. (15 points) Prove $\sum_{i=0}^{n}\binom{n}{i} = 2^n$ for $n \in Z^+$.
The Binomial Theorem states $\sum_{i=0}^{n} x^i y^{n-i}\binom{n}{i} = (x+y)^n$. Let $x = y = 1$ and we see that $\sum_{i=0}^{n}\binom{n}{i} = \sum_{i=0}^{n} 1^i 1^{n-i}\binom{n}{i} = (1+1)^n = 2^n$.

4. (15 points) Compute $\sum_{i=0}^{20} 2^i 3^{20-i}\binom{20}{i}$.
The Binomial Theorem states $\sum_{i=0}^{n} x^i y^{n-i}\binom{n}{i} = (x+y)^n$. Let $n = 20, x = 2$ and $y = 3$, and we see that $\sum_{i=0}^{20} 2^i 3^{20-i}\binom{20}{i} = 5^{20} = 95\,367\,431\,640\,625$.

5. (15 points) Use a combinatorial proof to show $\binom{n}{k} = \binom{n}{n-k}$ for all $n, k \in Z^+$ where $k \leq n$.
Selecting a subset of k items from n items is equivalent to selecting $n-k$ items to exclude from the subset.

6. (20 points) Use a combinatorial proof to show $\binom{4n}{2} = 4\binom{n}{2} + 6n^2$ for $n \in Z^+$.
Let $A = \{1, 2, ..., n, ..., 4n\}$ and let S be the collection of all subsets of A of size 2. On the one hand, it is clear that $|S| = \binom{4n}{2}$. One the other hand partition A into four disjoint subsets. Let $B = \{1, 2, ...n\}$, $C = \{n+1, n+2, ...2n\}$, $D = \{2n+1, 2n+2, ...3n\}$ and $E = \{3n+1, ..., 4n\}$. We can select a set of two elements of A by selecting two elements from one of the sets B, C, D, or E. This can be done in $4\binom{n}{2}$ ways. Or we can select two of the sets B, C, D, or E and select one element from each.

This can be done in $\binom{4}{2}n^2 = 6n^2$ ways. Thus, $|S| = 4\binom{n}{2} + 6n^2$. We've counted the same set S in two different ways and $\binom{4n}{2} = 4\binom{n}{2} + 6n^2$ for $n \in Z^+$.