Math 3322 Test I
DeMaio Fall 2008

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. Where appropriate, simplify all answers to a single decimal expansion.

Preliminaries

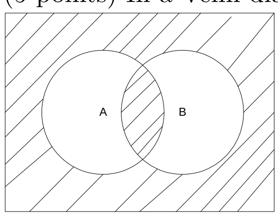
1. (5 points) List the members of the set $S = \left\{x|x \in Z^+, 50 \le x^3 \le 150\right\}$.
   $S = \{4, 5\}$

2. *Quiz 1 # 3* (5 points) Construct $P(A)$ for $A = \{*, a, 3\}$.
   $P(A) = \{\emptyset, \{*\}, \{a\}, \{3\}, \{*, a\}, \{*, 3\}, \{a, 3\}, \{*, a, 3\}\}$

3. *Quiz 1 # 4* (5 points) Compute $|P(A)|$ for $A = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31\}$.
   $|P(A)| = 2^{11} = 2048$

4. (5 points) Give an example of sets A and B such that B is a proper subset of A and $|A| = |B|$.
   Let $A = Z$ and let $B = Z^+$.

5. (5 points) In a Venn diagram, shade $\overline{A \oplus B}$.



6. *HW 2.2 #1* (5 points) Let A be the set of students who live within one mile of campus. Let B be the set of all students who walk to class. What does it mean to say Mary is a member of the set $\overline{A} \cap B$?
   Mary walks more than one mile to her classes on campus.

7. *HW 2.3 #8h* (5 points) Compute $\left\lfloor \frac{1}{2} + \left\lfloor \frac{1}{2} + \left\lceil \frac{1}{2} \right\rceil \right\rfloor \right\rfloor$.
   $\left\lfloor \frac{1}{2} + \left\lfloor \frac{1}{2} + \left\lceil \frac{1}{2} \right\rceil \right\rfloor \right\rfloor = 1$

8. *HW 2.3 #4a* (5 points) Find the domain and range of the function that assigns to each positive integer its last digit.
   The domain is Z^+. The range is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

9. *HW 2.3 #16d* (5 points) Give an example of a function $f : Z \to Z^+$ that is neither one-to-one nor onto.
   Let $f(n) = n^2$.

10. *HW 2.4 #23* (5 points) Compute $\sum_{i=50}^{175} i$.
    $\sum_{i=50}^{175} i = \sum_{i=1}^{175} i - \sum_{i=1}^{49} i = \frac{175*176}{2} - \frac{49*50}{2} = 14\,175$.

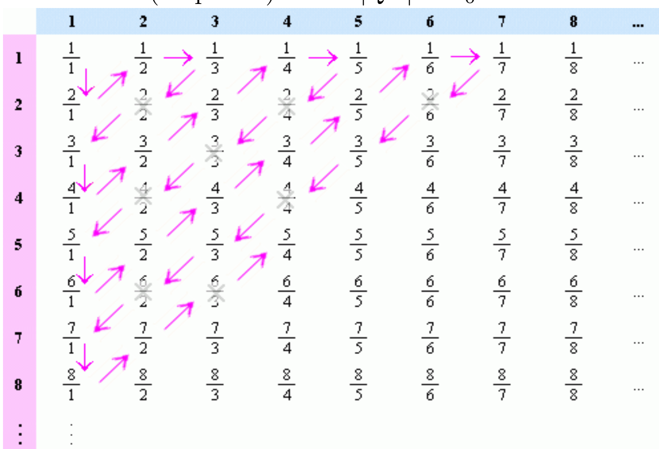
11. *Quiz 2 # 1d* (5 points) Compute $\prod_{-533}^{278} \left(i^3 - 1\right)$.
    Since 1 is an integer between -533 and 278 the product contains the term $1^3 - 1 = 0$ and is 0.

12. (5 points) Compute $\frac{100!}{95!5!}$.
    $\frac{100!}{95!5!} = \frac{100*99*98*97*96}{120} = 75\,287\,520$.

Problems

13. (10 points) True or False? If true, prove it. If false, provide a counter-example.
    $(j + k)! = j! + k!$
    False! Let $k = j = 2$. Then $(j + k)! = (2 + 2)! = 41 = 24$ while $2! + 2! = 2 + 2 = 4$.

14. *Class Notes* (10 points) Prove $|Q^+| = \aleph_0$.



from http://www.homeschoolmath.net/teaching/rational-numbers-countable.php.

15. *HW 4.1 #4* (10 points) Use mathematical induction to prove $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in Z^+$.

I. Show that $S(1)$ is true. L.H.S. $\sum_{i=1}^{1} i^3 = 1^3 = 1$. R.H.S. $\frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$. Thus, $S(1)$ is true.

II. Assume $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ and show $\sum_{i=1}^{n+1} i^3 = \frac{(n+1)^2(n+2)^2}{4}$.
$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3$ which by the inductive hypothesis is $\frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} = \frac{(n+1)^2\left[n^2+4(n+1)\right]}{4} = \frac{(n+1)^2\left[n^2+4n+4\right]}{4} = \frac{(n+1)^2(n+2)^2}{4}$.

16. (10 points) Use induction to prove 3 divides $n^3 + 2n - 15$ for all $n \in Z^+$.
I. $\frac{1^3+2*1-15}{3} = \frac{1+2-15}{3} = \frac{-12}{3} = -4 \in Z$.
II. Assume $\frac{n^3+2n-15}{3} \in Z$. Show $\frac{(n+1)^3+2(n+1)-15}{3} \in Z$
$\frac{(n+1)^3+2(n+1)-15}{3} = \frac{\left(n^3+3n^2+3n+1\right)+(2n+2)-15}{3} = \frac{n^3+2n-15}{3} + \frac{3n^2+3n+3}{3}$ which by the inductive assumption is $int + \frac{3n^2+3n+3}{3} = int + \frac{3\left(n^2+n+1\right)}{3} = int + n^2 + n + 1 = int$.

17. (10 points) Find the error in the following proof of this "theorem":

"Theorem: Every positive integer equals the next largest positive integer."

" Proof: Let $P(n)$ be the proposition '$n = n+1$'. To show that $P(k) = P(k+1)$, assume that $P(k)$ is true for some k, so that $k = k+1$. Add 1 to both sides of this equation to obtain $k+1 = k+2$, which is $P(k+1)$. Therefore $P(k) = P(k+1)$ is true. Hence $P(n)$ is true for all positive integers n.".
There is no base case to show that $1 = 2$.

18. *HW 2.2 #41* (10 points) For sets A, B and C, let $A \oplus C = B \oplus C$. Use contradiction to prove $A = B$.
Assume $A \neq B$. Without loss of generality we can say that if $A \neq B$, then there exists $x \in A$ such that $x \notin B$. We now have two cases.
1. If $x \in C$, then $x \notin A \oplus C$ but $x \in B \oplus C$. This is a contradiction since $A \oplus C = B \oplus C$.
2. If $x \notin C$, then $x \in A \oplus C$ but $x \notin B \oplus C$. This is a contradiction since $A \oplus C = B \oplus C$.
In either case a contradiction arises. Thus, the assumption is wrong and $A = B$.