Math 3322 Test II
DeMaio Fall 2008

Name____

Instructions. Show all your work. Credit cannot and will not be awarded for work not shown. Where appropriate, simplify all answers to a single decimal expansion.

1. (4 points each) Let S = {1, 2, 3, 4, 5, 6, 7}.
(a) How many non-empty subsets does S have? $2^7 - 1 = 127$

(b) How many subsets of S have no odd numbers as members? $2^3 = 8$

(c) How many subsets of S have exactly 4 elements? $\binom{7}{4} = 35$

(d) How many four digit numbers can be made using the digits of S if a digit may be used repeatedly? $7^4 = 2401$

(e) How many even four digit numbers can be made using the digits of S if a digit may be used only once? The last digit must be even so we will pick it first and then select the remaining digits. This can be done in $3 * 6 * 5 * 4 = 360$ ways.

2. (5 points each) Susan buys an economy pack of fifty pens. The pens are identical except for color. There are ten of each of five different colors.

(a) How many different ways can Susan select four pens of different colors to take to work? $\binom{5}{4} = 5$

(b) How many different ways can Susan select ten pens to take to work? $\binom{5+10-1}{10} = 1001$

(c) How many different ways can Susan select ten pens to take to work with at least one of each color? $\binom{5+5-1}{5} = 126$

3. (5 points) How many strings of 6 decimal digits have exactly three digits that are 9's? $\binom{6}{3} * 9^3 = 14\,580$

4. (5 points each) Jason's CD collection consists of five different rock CD's, three different jazz CD's, two different blues CD's, two different classical CD's and a single folk CD. Jason is planning a trip and randomly selects four CD's. How many different ways can this be done
(a) with no restrictions; $\binom{13}{4} = 715$
(b) by only selecting from the rock CD's; $\binom{5}{4} = 5$
(c) with no more than one CD from any genre? With 5 genres we are electing to select one cd from each genre with one genre excluded. This can be done in $3*2*2*1+5*2*2*1+5*3*2*1+5*3*2*1+5*3*2*2 = 152$ ways.

5. (10 points) How many positive integers not exceeding 1000 are divisible by 6 or 9? $\lfloor \frac{1000}{6} \rfloor + \lfloor \frac{1000}{9} \rfloor - \lfloor \frac{1000}{18} \rfloor = 222$

6. (10 points) Use the binomial theorem to expand $(3x-2)^4$ into polynomial form. You must show all details.
$(3x-2)^4 = \binom{4}{4}(3x)^4(-2)^{4-4} + \binom{4}{3}(3x)^3(-2)^{4-3} + \binom{4}{2}(3x)^2(-2)^{4-2} + \binom{4}{1}(3x)^1(-2)^{4-1} + \binom{4}{0}(3x)^0(-2)^{4-0} = 81x^4 - 216x^3 + 216x^2 - 96x + 16$

7. (5 points each) Find the coefficient of x^{10} in the expansion of
(a) $(2x-4)^{15}$; $\binom{15}{10}(2x)^{10}(-4)^5 = -3148\,873\,728x^{10}$
(b) $(4x^3-3)^{12}$; The coefficient is 0 since no integer power of $4x^3$ will yield x^{10}.
(c) $(8x^5-3)^5$. $\binom{5}{2}(8x^5)^2(-3)^3 = -17\,280x^{10}$

8. (10 points) Use the Pigeonhole Principle to show that if seven **distinct** numbers are selected from $\{1, 2, ..., 11\}$, then some two of these numbers sum to 12. For an extra 5 bonus points instead show that if $n+1$ numbers are selected from $\{1, 2, ..., 2n-1\}$, then some two of these numbers sum to $2n$. In either case you must carefully describe the pigeonholes and how the pigeons are placed.
We will create n pigeonholes and each pigeonhole will have two numeric labels:
$\lfloor 1, 2n-1 \rfloor, \lfloor 2, 2n-2 \rfloor, \lfloor 3, 2n-3 \rfloor, ..., \lfloor n-1, n+1 \rfloor, \lfloor n \rfloor$.

Note that the sum of the two labels for a box is $2n$. We place the selected numbers in the pigeonhole that contains its label. We have placed $n+1$ numbers into the n pigeonholes and at least one pigeonhole contains at least two pigeons. Those two numbers sum to $2n$.

9. (10 points) Prove $\sum_{i=0}^{n} \binom{n}{i} = 2^n$. The Binomial Theorem states $\sum_{i=0}^{n} x^i y^{n-i} \binom{n}{i} = (x+y)^n$. Let $x = y = 1$ and we see that $\sum_{i=0}^{n} \binom{n}{i} = \sum_{i=0}^{n} 1^i 1^{n-i} \binom{n}{i} = (1+1)^n = 2^n$.

10. (10 points) Use a combinatorial proof to show $\binom{4n}{2} = 4\binom{n}{2} + 6n^2$ for $n \in Z^+$.
Let $A = \{1, 2, ..., n, ..., 4n\}$ and let S be the collection of all subsets of A of size 2. On the one hand, it is clear that $|S| = \binom{4n}{2}$. One the other hand partition A into four disjoint subsets. Let $B = \{1, 2, ...n\}$, $C = \{n+1, n+2, ...2n\}$, $D = \{2n+1, 2n+2, ...3n\}$ and $E = \{3n+1, ..., 4n\}$. We can select a set of two elements of A by selecting two elements from one of the sets B, C, D, or E. This can be done in $4\binom{n}{2}$ ways. Or we can select two of the sets B, C, D, or E and select one element from each. This can be done in $\binom{4}{2}n^2 = 6n^2$ ways. Thus, $|S| = 4\binom{n}{2} + 6n^2$. We've counted the same set S in two different ways and $\binom{4n}{2} = 4\binom{n}{2} + 6n^2$ for $n \in Z^+$.