1 A Brief Introduction to the Mathematics of Chess Puzzles

Puzzles on the chessboard have long been studied by mathematicians. Naturally, we do not restrict ourselves only to the standard 8×8 chessboard. Generalizations are quickly made to the square board, the rectangular board, etc. We will concentrate on three types of problems.

2 The Knight's Tour

The closed knight's tour of a chessboard is a classic problem in mathematics. Can the knight use legal moves to visit every square on the board and return to its starting position? When translated into graph theoretic terms, this is equivalent to the existence of a Hamiltonian cycle. The unique movement of the knight makes a tour an interesting existence problem. For other chess pieces, tours are a trivial existence problem.

Moves of the Knight

38	35	62	25	60	23	10	7
63	26	37	34	11	8	59	22
36	39	28	61	24	57	6	9
27	64	33	40	5	12	21	58
50	29	4	13	48	41	56	19
1	14	49	32	53	20	47	44
30	51	16	3	42	45	18	55
15	2	31	52	17	54	43	46

A Closed Knight's Tour of the 8×8 Chessboard

While originally studied for the standard board, the problem is easily generalized to rectangular boards.

1	4	7	26	13	28	11	22	19	16
6	25	2	29	8	23	14	17	10	21
3	30	5	24	27	12	9	20	15	18

A Closed Knight's Tour of the 3×10 Chessboard

1	22	27	12	7	16
28	11	30	15	26	13
21	2	23	8	17	6
10	29	4	19	14	25
3	20	9	24	5	18

A Closed Knight's Tour of the 5×6
Chessboard

In 1991 Schwenk [1] completely answered the question: Which rectangular chessboards have a knight's tour?

Theorem 1 (Schwenk) *An $m \times n$ chessboard with $m \leq n$ has a closed knight's tour unless one or more of the following three conditions hold:*

(a) m and n are both odd
(b) $m \in \{1, 2, 4\}$;
(c) $m = 3$ and $n \in \{4, 6, 8\}$.

Clearly, no tours can exist for the or boards.

For the 3×4 board consider the squares with only two possible moves.

1	4	7	10
2	5	8	11
3	6	9	12

The 3×4 Chessboard

Use of these squares forms two closed cycles and no closed tour may exist. The cases for the 3×6 and 3×8 boards are similar and left as an exercise.

For m and n both odd note that a legal move for a knight whose initial position is a white square will always result in an ending position on a black square and vice-versa. Hence, every graph representing the legal moves of a knight is bipartite.

A necessary condition for the existence of a Hamiltonian cycle in a bipartite graph $G = (X \cup Y, E)$ is that $|X| = |Y|$. However if m and n are both odd then there will exist an odd number of squares on the board and the number of black squares will not equal the number of white squares. Thus, no closed tour can exist.

For $4 \times n$ boards we will use a different coloring of the board.

Coloring the $4 \times n$ Chessboard

Note that a knight must move from a red square to a yellow square and a knight must move from a purple square to a green square. Two closed cycles are now forced and no closed tour exists for the $4 \times n$ board.

This covers all the rectangular boards that do not admit a closed tour.

Proof of the existence of a tour of all other boards is by induction. Constructions are provided that yield a tour for all other boards.

For example a closed 3×10 tour and an open 3×4 tour are used to construct a closed 3×14 tour.

4

1	4	7	26	13	28	11	22	19	16
6	25	2	29	8	23	14	17	10	21
3	30	5	24	27	12	9	20	15	18

a	d	g	j
l	i	b	e
c	f	k	h

Constructing a Closed Knight's Tour of the 3×14
Chessboard

3   The Domination Problem

Given an $n \times n$ board, find the domination number, which is the minimum number of queens (or other pieces) needed to attack or occupy every square.

Domination of the 8×8 Chessboard
with 5 Queens

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\gamma(Q_n)$	1	1	1	2	3	3	4	5	5	5	5	6	7	8	9	9	9	9	10

n	20	21	22	23	24	25
$\gamma(Q_n)$	10 or 11	11	11 or 12	12	12 or 13	13

Known Domination Numbers

4   The Independence Problem

The eight queens puzzle is the problem of putting eight chess queens on an 8×8 chessboard such that none of them is able to capture any other using the standard chess queen's moves. The colour of the queens is meaningless in this puzzle, and any queen is assumed to be able to attack any other. Thus, a solution requires that no two queens share the same row, column, or diagonal. The eight queens puzzle is an example of the more general n queens puzzle of placing n queens on an $n \times n$ chessboard.

Eight non-taking Queens on
the 8×8 Chessboard

For $n \geq 4$, there always exists an arrangement of n queens on an $n \times n$ board. The proof of this fact provides constructions for the different types of boards.

1,2

2,2

3,1

$_1 4.jpg$

2,1

$_1 5.jpg$

References

[1] A. J. Schwenk, Which Rectangular Chessboards have a Knight's Tour? *Mathematics Magazine* 64:5 (December 1991) 325-332.

[2] J. J. Watkins, *Across the Board: The Mathematics of Chessboard Problems*, Princeton University Press, Princeton, 2004.