

## 4.2 Chromatic Polynomial

The seven coworkers who were organizing a lunch excursion in the previous section required at least three cars to go to lunch given their restrictions. One such solution appeared in Figure 4.1.2. A second three car solution was also given in section 4.1. A natural question is how many different 3-colorings (car assignments) are possible for the graph. Of course, listing all possible colorings is one way to answer this question, but such enumeration is subject to error especially when the number becomes large. Sometimes by using familiar counting techniques a polynomial may be found which will generate this answer.

Introduced by Birkhoff and Whitney in the 1930's (in an attempt to prove the Four Color Problem), the **chromatic polynomial of a graph  $G$** ,  $P(G, x)$ , is a polynomial function whose input is a non-negative integer number of colors  $x$  and whose output is the number of different legal colorings of a labeled graph  $G$  using up to and including  $x$  colors. For example, the chromatic polynomial of  $K_3$  is  $P(K_3, x) = x^3 - 3x^2 + 2x$ . Evaluating the function for  $x = 3$ , there are  $P(K_3, 3) = 3^3 - 3(3^2) + 2(3) = 6$  different legal colorings of  $K_3$  using up to three different colors. All six colorings appear in Figure 4.2.1. Since there are no legal colorings of  $K_3$  using one or two colors,  $P(K_3, 2) = P(K_3, 1) = 0$ , these are all possible 3-colorings for  $K_3$ . It is important to note that the coloring of the different labels is what differentiates the colorings from one another. There is only one way to color an unlabeled  $K_3$  with three colors. For  $x = 4$ ,  $P(K_3, 4) = 4^3 - 3(4^2) + 2(4) = 24$  gives the number of different legal colorings of  $K_3$  using up to four different colors. Because  $P(K_3, 3)$  gives the number of legal colorings of  $K_3$  using up to three different colors,  $P(K_3, 4) - P(K_3, 3) = 24 - 6 = 18$  gives the number of legal colorings using exactly four colors.

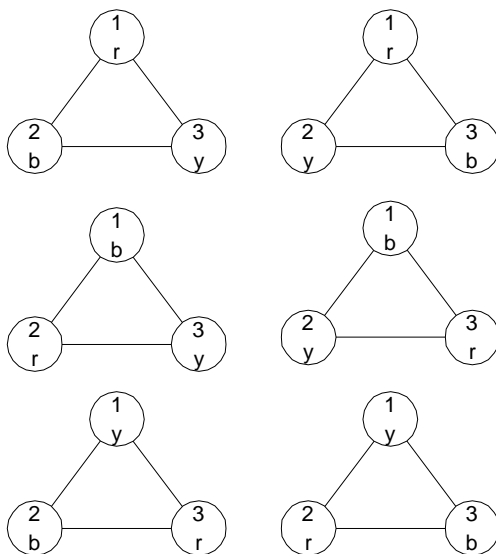


Figure 4.2.1

As can be seen, if the chromatic polynomial is known for a graph, the chromatic number can be found. This was the original intent when the polynomial was introduced but, of course, this is a big “if”. Finding the chromatic polynomial for a given graph can be quite difficult. The remainder of this section demonstrates the two different methods used to construct the chromatic polynomial of a graph. The first method is a straightforward application of the multiplication rule. Unfortunately, the multiplication rule may not always be adequate, but when it is, it is very fast and efficient.

Let’s use the multiplication rule to construct  $P(K_3, x)$ . If there are  $x$  colors available when coloring  $K_3$ , then there are  $x$  choices for the color of vertex 1. To color vertex 2, any color other than the color used for vertex 1 may be used giving  $x - 1$  color choices possible for vertex 2. For vertex 3, it is not permissible to use the colors of vertices 1 and 2, *which necessarily must be different from one another because they are adjacent vertices*. Thus, there are  $x - 2$  choices for coloring vertex 3. Applying the multiplication rule produces  $P(K_3, x) = x(x - 1)(x - 2) = x^3 - 3x^2 + 2x$ . This process is illustrated in Figure 4.2.2 where the work progresses from the upper left corner to the lower right corner. Recalling that the multiplication rule usually introduces order, it is clear why the graphs must be labeled for the technique to work.

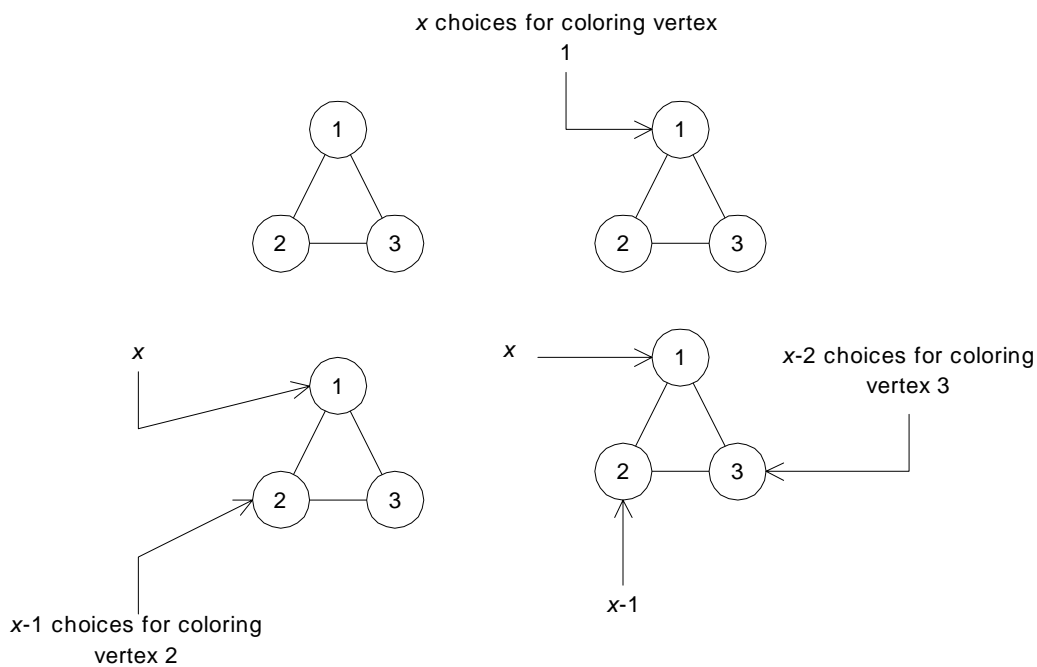


Figure 4.2.2

Another example appears in Figure 4.2.3. The method takes advantage of the fact that the vertices 1, 2 and 4 form a  $K_3$ . Hence, there are  $x$ ,  $x - 1$ , and  $x - 2$  choices respectively for the coloring of these vertices. Finally, vertex 3 is adjacent to the already colored vertices 1 and 4. Vertices 1 and 4 are adjacent and received different colors. Thus, there are  $x - 2$  different color choices for vertex 3 and  $P(G, x) = x(x - 1)(x - 2)^2$ . In each case, remember the variable label on

a vertex represents the number of color choices available for that vertex and not a specific color. Vertices 3 and 4 are not labeled with color  $x - 2$ . Vertices 3 and 4 both have  $x - 2$  color choices available at the time they are colored.

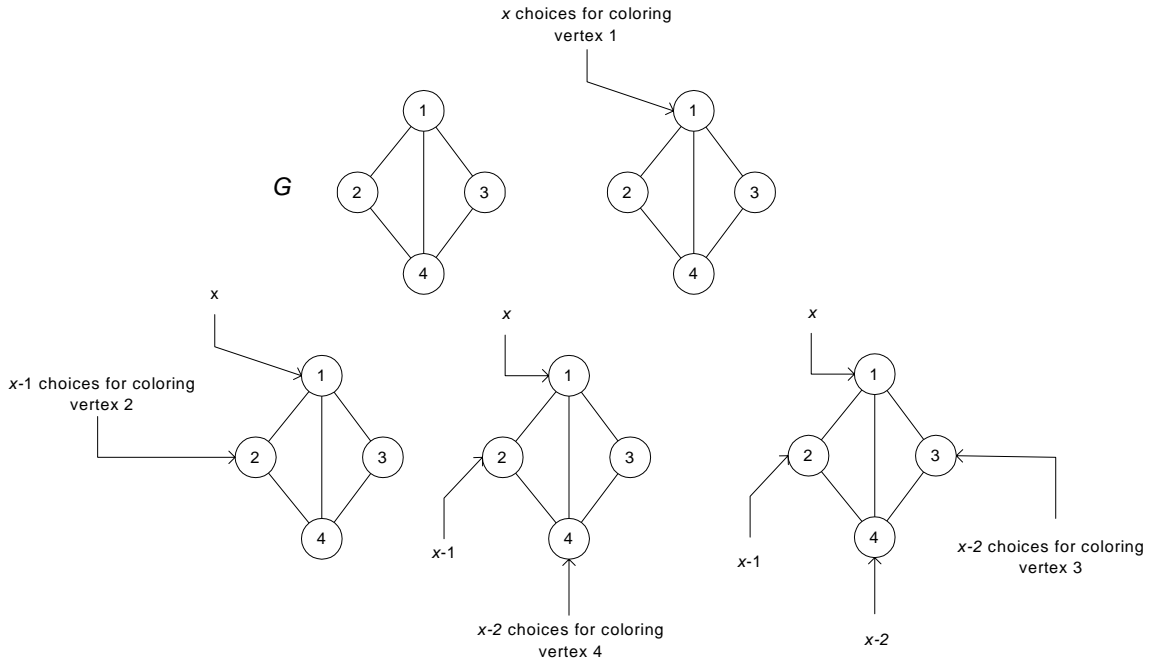


Figure 4.2.3

As previously stated, the multiplication rule may not be sufficient for determining the chromatic polynomial of a graph. Consider the easy to color cycle  $C_4$  seen in Figure 4.2.4. Determining  $P(C_4, x)$  with only the multiplication rule is impossible. When coloring vertices 1, 2 and 3, there are  $x$ ,  $x - 1$  and  $x - 1$  color choices respectively. The difficulty arises in coloring the final vertex. Vertex 4 cannot receive the same color as vertices 1 and 3, but these two vertices are not adjacent and could receive the same color or different. Hence, there are either  $x - 1$  or  $x - 2$  choices for coloring vertex 4. A simple application of the multiplication rule method is not possible, but the ‘or’ in this statement should remind the careful reader of the sum rule.

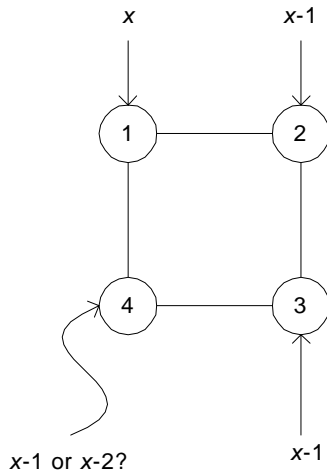


Figure 4.2.4

Two cases arise to use the sum rule. First, suppose vertices 1 and 3 receive the same color. Then there are  $x$  choices for coloring vertex 1,  $x - 1$  choices for coloring vertex 2, 1 choice for coloring vertex 3 (the same color as vertex 1) and  $x - 1$  choices for coloring vertex 4. The multiplication rule gives the polynomial  $x(x - 1)^2 = x^3 - 2x^2 + x$ . Second, suppose vertices 1 and 3 receive different colors. Then there are  $x$  choices for coloring vertex 1,  $x - 1$  choices for coloring vertex 2,  $x - 2$  choices for coloring vertex 3 (any color except those two different colors used for vertices 1 and 2) and  $x - 2$  choices for coloring vertex 4 (any color except those two different colors used for vertices 1 and 3). This gives the polynomial  $x(x - 1)(x - 2)^2 = x^4 - 5x^3 + 8x^2 - 4x$ . By the sum rule,  $P(C_4, x) = (x^3 - 2x^2 + x) + (x^4 - 5x^3 + 8x^2 - 4x) = x^4 - 4x^3 + 6x^2 - 3x$ .

Of course,  $C_4$  is easily analyzed in this manner. Can this idea of coloring nonadjacent vertices the same or different colors be used in more involved settings such as the lunch outing example from Section 4.1? The answer is “maybe”, because it depends on the graph. This idea leads to the second method called the reduction method. Before stating the method in Theorem 4.2.1, two new graph constructions must be defined.

The first construction formalizes the removal of an edge from a graph. Let  $G = (V, E)$  be a graph and  $ab$  an edge in  $G$ . The graph  $G_1$  is the resulting graph when  $ab$  is removed from  $G$ . Formally,  $G_1 = (V, E \setminus ab)$ . Sometimes this is informally written as  $G_1 = G \setminus ab$ .

The second construction defines how two vertices can be fused into one. The graph  $G_2$  results from collapsing the vertices  $a$  and  $b$  into a new vertex called  $*$ . Formally,  $G_2 = (V^*, E^*)$  where  $V^* = V \setminus \{a, b\} \cup \{*\}$  and  $E^* = E \setminus \{ac, bc \mid c \text{ is adjacent to either } a \text{ or } b \text{ in } G\} \cup \{*c \mid c \text{ is adjacent to either } a \text{ or } b \text{ in } G\}$ .

In other words, the vertex  $*$  now represents both vertices  $a$  and  $b$ , has all the same adjacencies  $a$  and  $b$  had in  $G$ , but has no multiple edges.

In the reduction method, the graph  $G_1$  represents the case where the vertices  $a$  and  $b$  are not adjacent so that they may be colored the same or different colors in a coloring of  $G_1$ . The graph  $G_2$  represents the case where the vertices  $a$  and  $b$  are colored the same color. Finally, the graph  $G$ , where  $a$  and  $b$  are adjacent, represents the case where  $a$  and  $b$  must be colored different colors. Returning to the construction of the chromatic polynomial for the cycle on four vertices, the above comments say  $C_4 = G_1$ . Why? The pair of vertices 1 and 3 are not adjacent in  $C_4$  and cause the problem of how to color vertex 4 because the pair could be colored the same color or a different color. The graph where vertices 1 and 3 are colored the same is  $G_2$  and  $P(G_2, x) = x^3 - 2x^2 + x$ . The graph where vertices 1 and 3 are colored differently is  $G$  and  $P(G, x) = x^4 - 5x^3 + 8x^2 - 4x$ . Thus,  $P(C_4, x) = P(G_1, x) = P(G_2, x) + P(G, x)$ . This suggests the following theorem due to Birkhoff and Lewis in 1946. Note, the  $C_4$  example will seem backwards once you see Theorem 4.2.1 applied. This is because the example was used to motivate the theorem, but it did not use the reduction method of the theorem to find  $P(C_4, x)$ .

**Theorem 4.2.1: Chromatic Polynomial Reduction Formula:** For any graph  $G = (V, E)$  and edge  $ab \in E$ ,  $P(G, x) = P(G_1, x) - P(G_2, x)$ .

**Proof:** Let  $G = (V, E)$  be a graph and  $ab$  an edge in  $G$ . The statement of the reduction formula is equivalent to  $P(G, x) + P(G_2, x) = P(G_1, x)$ . Graph  $G_1$  is the graph where vertices  $a$  and  $b$  may receive the same color or different colors. Graph  $G$  is simply the graph  $G_1$  where the edge  $ab$  is added and the vertices  $a$  and  $b$  receive different colors. Graph  $G_2$  is the graph  $G_1$  where vertices  $a$  and  $b$  are collapsed onto one another which is equivalent to vertices  $a$  and  $b$  receiving the same color in  $G_1$ . By the sum rule,  $P(G, x) + P(G_2, x) = P(G_1, x)$  and the reduction formula immediately follows.  $\square$

Although Theorem 4.2.1 is motivated and proven by the sum rule, it is written in a subtraction form and called a reduction formula because of the way it is applied. If it is difficult to find  $P(G, x)$  for the original graph  $G$ , it is reduced to  $G_1$  and  $G_2$  in hopes that  $P(G_1, x)$  and  $P(G_2, x)$  may be easily determined. Applying the reduction theorem to  $C_4$  and the 1 – 2 in Figure 4.2.5 yields two graphs whose chromatic polynomials may be easily determined using the multiplication rule.

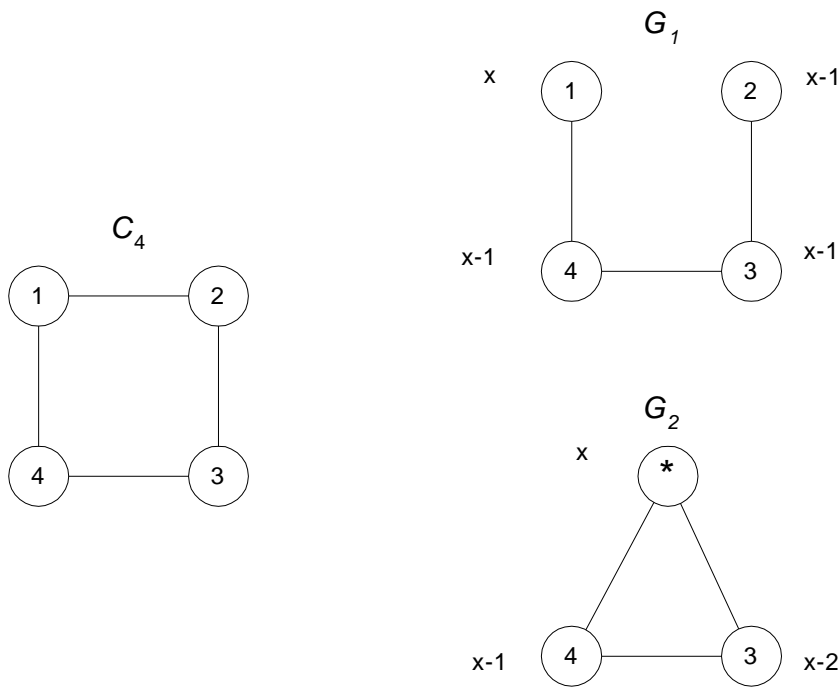


Figure 4.2.5

Thus,  $P(C_4, x) = P(G_1, x) - P(G_2, x) = x(x-1)^3 - x(x-1)(x-2) = x^4 - 4x^3 + 6x^2 - 3x$ . Though a reduction formula was used in this instance, the chromatic polynomial remains the same.

Let's use the reduction method to determine the chromatic polynomial for the graph in Figure 4.1.1. This graph, which will be  $G$ , is reproduced in Figure 4.2.6 with two vertices identified as  $a$  and  $b$  for the reduction. How are the two vertices selected? As you may have observed in the previous examples in this section, having subgraphs that are  $K_3$  make the counting with the multiplication rule easy. Thus,  $a$  and  $b$  were chosen such that when the vertex  $*$  is created there will be more triangles in  $G_2$  than originally in  $G$ .

Figure 4.2.6 also shows  $G_1$  and  $G_2$  (with vertex  $*$  identified) for the reduction method. The multiplication rule method can be applied to  $G_2$  easily and the vertices are labeled with the possible color choices for each vertex. In this case, two  $K_4$  subgraphs aid in the counting because the four vertices must be colored different colors. This gives  $P(G_2, x) = x(x-1)(x-2)^2(x-3)^2$ .

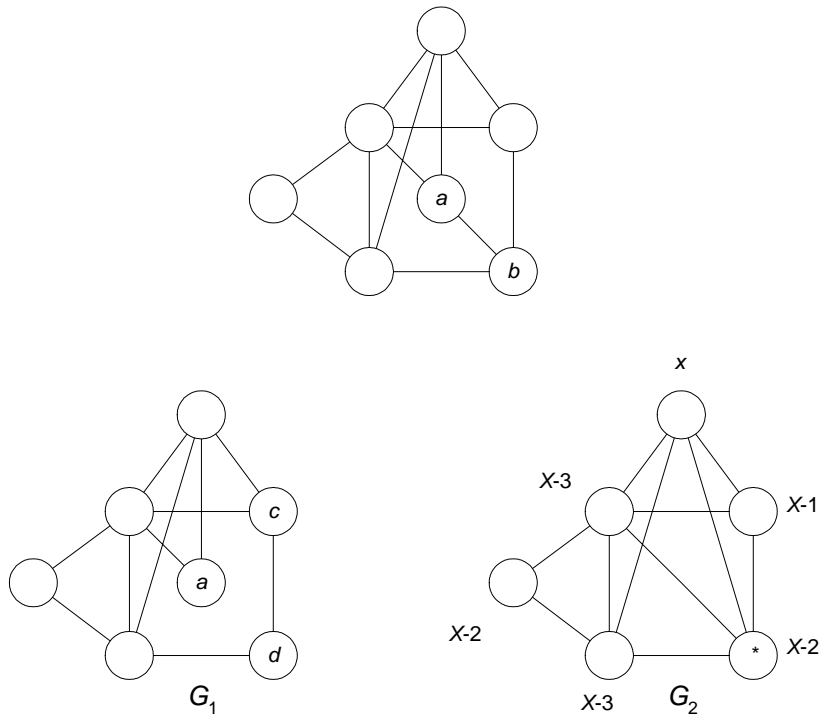


Figure 4.2.6

On the other hand, the cycle of length four in  $G_1$  creates a problem similar to the situation for  $C_4$  alone. A second reduction is performed on  $G_1$  using the vertices labeled  $c$  and  $d$  in Figure 4.2.6. The two resultant graphs  $H_1$  and  $H_2$  appear in Figure 4.2.7.

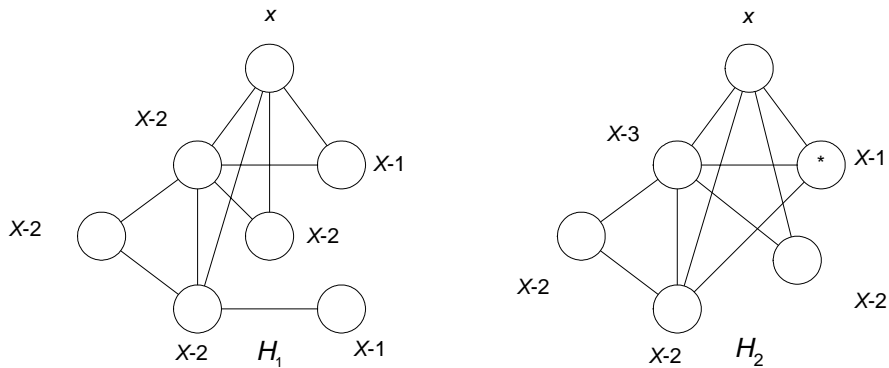


Figure 4.2.7

Because of the structures of  $H_1$  and  $H_2$ , the multiplication rule may be used to determine their chromatic polynomials. Once again, the vertices are labeled with the possible choices of colors. The two resultant polynomials are  $P(H_1, x) = x(x-1)^2(x-2)^4$  and

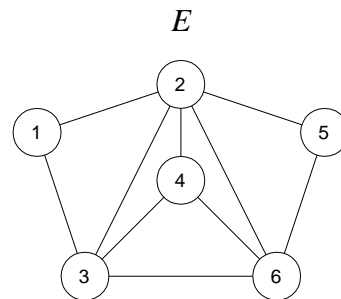
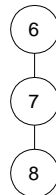
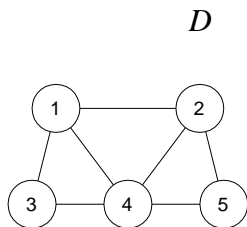
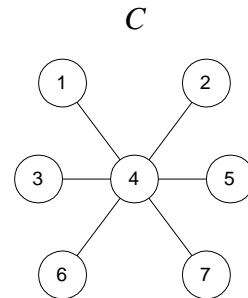
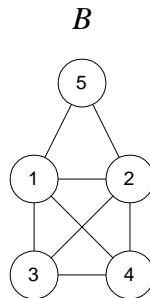
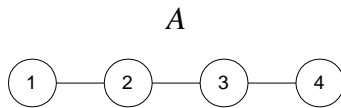
$P(H_2, x) = x(x-1)(x-2)^3(x-3)$ . Using Theorem 4.2.1 gives  
 $P(G, x) = P(G_1, x) - P(G_2, x) = [P(H_1, x) - P(H_2, x)] - P(G_2, x)$ . After much simplifying the chromatic polynomial is found to be  $P(G, x) = x^7 - 12x^6 + 62x^5 - 174x^4 + 275x^3 - 228x^2 + 76x$ . With the chromatic polynomial available the question asked at the beginning of the section may be answered. There are  $P(G, 3) = 12$  different 3-colorings (car assignments) for the problem.

There are many interesting facts associated with the chromatic polynomial. You may have noticed that the degree of the polynomial (the highest power in its expansion) is the number of vertices  $n$ . The leading coefficient (the coefficient on the  $x^n$  term) is 1. The coefficients of a chromatic polynomial alternate in sign (from positive to negative). The constant term is zero. And the absolute value of the coefficient on the  $x^{n-1}$  term is the number of edges. In fact, formulas are known for determining the coefficients of the chromatic polynomials that use the principle of inclusion and exclusion that appears later in the text.

With so much known about the chromatic polynomial, why is it difficult to find? The answer returns to the fact that computing the chromatic polynomial is in the class of NP-complete problems just like the chromatic number. There are no efficient algorithms that work on all types of graphs. The next section demonstrates three polynomial time algorithms for finding the chromatic number.

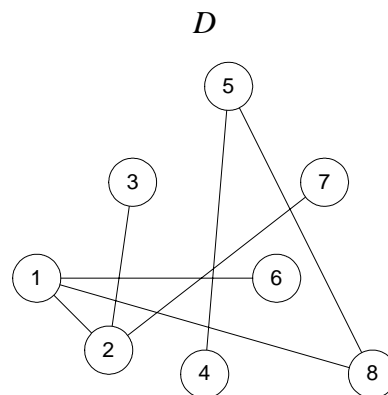
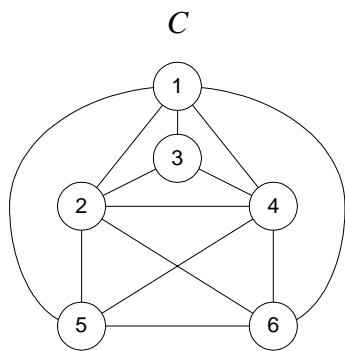
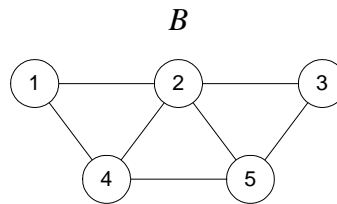
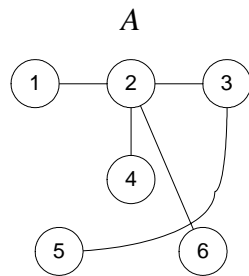
### Homework

- Determine the chromatic polynomial for each of the following graphs.



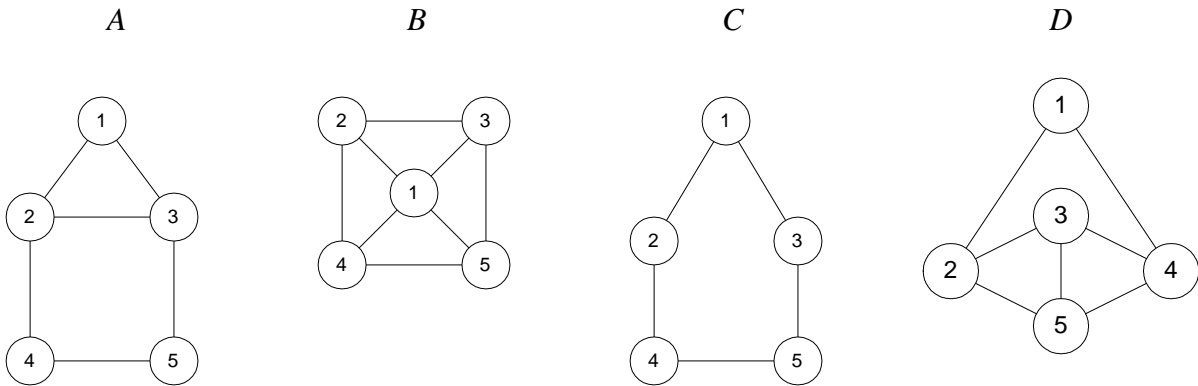


2. Determine the chromatic polynomial for each of the following graphs.

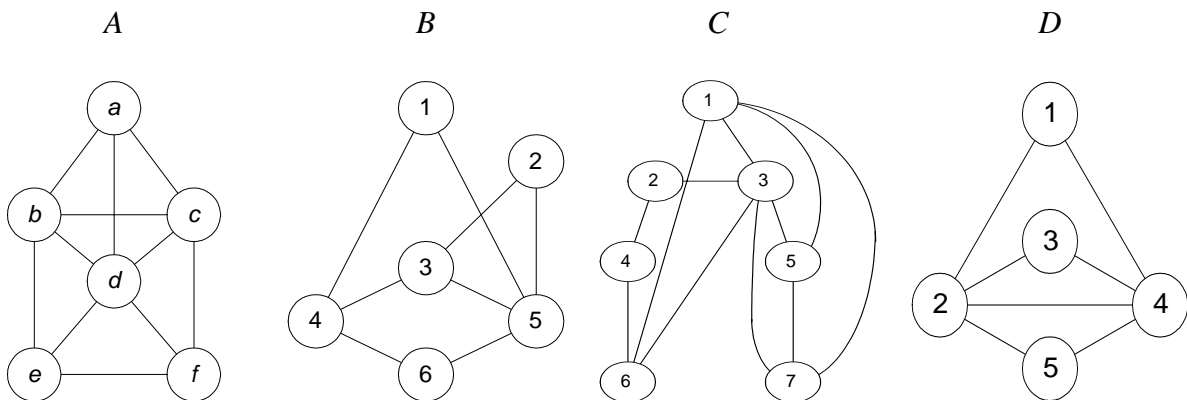


3. For each of the graphs in Exercise 1, find the
- i. number of different legal colorings using at most three colors;
  - ii. chromatic number.
4. For each of the graphs in Exercise 2, find the
- i. number of different legal colorings using at most three colors;
  - ii. chromatic number.
5. Find  $P(K_n, x)$ .
6. Find  $P(N_n, x)$ .

7. Using the chromatic polynomial reduction formula, determine the chromatic polynomial for each of the following graphs.



8. Using the chromatic polynomial reduction formula, determine the chromatic polynomial for each of the following graphs.



9. Siblings Joe, Mary, and Nick are planning a vacation retreat with their children. Joe has one child Lisa. Mary has three children, Donna, Michael and Richard. Nick has two children, Danielle and Michelle. The rented house has five different children's bedrooms that can contain at most three children each. It has been decided that boys and girls will not share a room nor will siblings share a room. If not all five bedrooms must be utilized, how many different ways can the children's sleeping arrangements be made?
10. If all five bedrooms in the previous problem must be utilized, how many different ways can the children's sleeping arrangements be made?
11. Prove that the constant term in every chromatic polynomial is 0. Hint! Consider the number of legal colorings with no colors.

12. Let  $G = (V, E)$  be a graph with at least one edge. Prove that the sum of the coefficients of  $P(G, x)$  is 0.
13. If  $P(G, x) = x(x-1)^5(x-2)^3$  then  $\chi(G) =$ \_\_\_\_\_.
14. If  $P(G, x) = x(x-1)^7 - x(x-1)^5(x-2)$  then  $\chi(G) =$ \_\_\_\_\_.
15. Construct an example of a graph  $G$  such that
  - i.  $\chi(G) < \chi(G_2)$ ;
  - ii.  $\chi(G) > \chi(G_2)$ .
16. Construct an example of a graph  $G$  such that
  - i.  $\omega(G) < \omega(G_2)$ ;
  - ii.  $\omega(G) > \omega(G_2)$ .
17. Consider all the labeled trees on four vertices. Find the chromatic polynomial for each.
18. Let  $T$  be a tree. Find  $P(T, x)$ .
19. Using the result in Question 18, can two non-isomorphic graphs have the same chromatic polynomial?